AUTOMATED MULTI-LEVEL SUB-STRUCTURING FOR FLUID-SOLID INTERACTION PROBLEMS

MARKUS STAMMBERGER AND HEINRICH VOSS*

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Abstract. The Automated Multi-Level Sub-structuring (AMLS) method is a powerful technique to determine a large number of eigenpairs with moderate accuracy of huge symmetric and definite eigenvalue problems in structural analysis. This paper is concerned with an adapted version of AMLS for eigenfrequency analysis of fluid-solid interaction systems. Although fluid-solid vibrations are governed by an unsymmetric eigenproblem the modified AMLS method needs approximately the same computational effort. An error bound related to the eigenvalue approximations is proved.

1. Introduction. Over the last few years, a new method for huge linear eigenvalue problems

$$Kx = \lambda Mx \tag{1.1}$$

where $K \in \mathbb{C}^{n\times n}$ and $M \in \mathbb{C}^{n\times n}$ are Hermitian and positive definite, known as *Automated Multi–Level Sub-structuring (AMLS)*, has been developed by Bennighof and co-authors, and has been applied to frequency response analysis of complex structures [3, 5, 6, 7]. Here the large finite element model is recursively divided into very many sub-structures on several levels based on the sparsity structure of the system matrices. Assuming that the interior degrees of freedom of sub-structures depend quasistatically on the interface degrees of freedom, and modelling the deviation from quasistatic dependence in terms of a small number of selected sub-structure eigenmodes the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest. Recent studies ([18, 21, 28], e.g.) in vibro-acoustic analysis of passenger car bodies, where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed, have shown that for this type of problems AMLS is considerably faster than Lanczos type approaches.

On each level of the hierarchical sub-structuring AMLS consists of two steps. First for every sub-structure of the current level a congruence transformation is applied to the matrix pencil to decouple in the stiffness matrix the sub-structure from the degrees of freedom of higher levels. Secondly, the dimension of the problem is reduced by modal truncation of the corresponding diagonal blocks discarding eigenmodes according to eigenfrequencies which exceed a predetermined cut-off frequency. Hence, AMLS is nothing else but a projection method where the large problem under consideration is projected to a search space spanned by a smaller number of eigenmodes of clamped sub-structures on several levels.

In this paper we consider free vibrations of an elastic structure which is completely filled with a homogeneous, inviscid and compressible fluid (liquid or gas). Different formulations have been proposed to model this problem ([23, 24, 25, 30], e.g.), the most obvious of which describes the structure by its relative displacement field and the fluid by its pressure [26, 30]. Such an approach leads to an unsymmetric eigenvalue problem which after discretization obtains the following form.

$$Kx := \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ -C^T & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} =: \lambda Mx. \tag{1.2}$$

Here the stiffness and mass matrices $K_s, M_s \in \mathbb{R}^{s\times s}$ of the structure and the mass matrix $M_f \in \mathbb{R}^{f\times f}$ of the fluid are symmetric and positive definite, and the stiffness matrix of the fluid $K_f \in \mathbb{R}^{f\times f}$ is symmetric positive semi-definite. If K_f is singular, we assume that $\mathrm{Ker}(K_f) =$

*Institute of Numerical Simulation, Hamburg University of Technology, D-21071 Hamburg, Germany ({markus.stammberger,voss}@tu-harburg.de)

span$\{(1,\dots,1)^T\}$ as this case results from a Finite Element discretization with a nodal basis and Neumann boundary conditions. $C \in \mathbb{R}^{s\times f}$ describes the coupling between the solid and the fluid.

Several authors [1, 8, 9, 10, 12] emphasize that the unsymmetry of problem (1.2) makes it inconvenient from the numerical point of view. However, (1.2) can be shown to have real non-negative eigenvalues which can be characterized as minmax values of a Rayleigh functional [27] and therefore numerical methods such as structure preserving iterative projection methods exist which have similar properties as the ones for symmetric problems.

In industrial praxis (1.2) is often solved in the following way (cf. [4, 18, 21, 28], e.g.). Neglecting the coupling between the fluid and the structure one first applies an efficient solver like the shift-and-invert Lanczos method or AMLS to the symmetric eigenvalue problems

$$K_s v_s = \omega_s M_s v_s \quad \text{and} \quad K_f v_f = \omega_f M_f v_f. \tag{1.3}$$

Afterwards the original problem is projected to the space spanned by $V = \operatorname{diag}\{V_s, V_f\}$ where $V_s := [v_{s1},\dots,v_{sk}]$ and $V_f := [v_{f1},\dots,v_{f\ell}]$ contain the eigenmodes of (1.3) corresponding to eigenfrequencies not exceeding a certain cut-off level. The projected problem

$$\begin{pmatrix} V_s^T K_s V_s & V_s^T C V_f \\ 0 & V_f^T K_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix} = \lambda \begin{pmatrix} V_s^T M_s V_s & 0 \\ -V_f^T C^T V_s & V_f^T M_f V_f \end{pmatrix} \begin{pmatrix} y_s \\ y_f \end{pmatrix} \tag{1.4}$$

then yields approximations for the original eigenvalue problem (1.2).

This approach does not involve the coupling terms in the construction of the ansatz space span$\{V\}$, and the approximation quality is only satisfying if the coupling is weak (like the coupling of the body of a car and the enclosed air), for strongly coupled systems the relative errors can become arbitrarily large as it can be seen from the following low-dimensional example.

Consider

$$\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix} x = \lambda \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} x, \tag{1.5}$$

where $a \in \mathbb{R}$ is the coupling term.

For $a \to \infty$ the eigenvalues tend to 0 and ∞. Applying AMLS as done in industrial applications with cut-off frequency $\omega = 1.5$, we obtain $V = e_2$ and the approximate eigenvalue is 1 independently of a whereas for large a the exact smallest eigenvalue is close to zero.

In this paper we propose a version of AMLS for the unsymmetric eigenvalue problem (1.2) that incorporates the coupling term into the reduction process. We take advantage of a symmetric eigenvalue problem of double dimension such that the eigensolutions of problem (1.2) can be recovered from the ones of the symmetric problem. It is noteworthy that the derived AMLS reduction requires essentially the same cost as the AMLS method for a symmetric problem of the dimension of (1.2).

The paper is organized as follows. Section 2 recalls the AMLS method for symmetric definite eigenvalue problems, and Section 3 describes how to generalize AMLS to the fluid-solid vibration problem such that the coupling is incorporated into the reduction, and it discusses its algorithmic efficiency. In Section 4 we derive an a priori bound to the relative errors of the eigenvalue approximations by AMLS which can not be improved without further assumptions, and Section 5 contains a numerical example which demonstrates the improvement of our approach.

2. AMLS for symmetric eigenvalue problems. This section reviews the Automated Multi-Level Sub-structuring (AMLS) method for solving the eigenproblem

$$Kx = \lambda M x \tag{2.1}$$

where K and M are sparse, symmetric and positive definite matrices. AMLS was introduced by Bennighof [3, 7]. It is a multilevel extension of the Component Mode Synthesis method developed by Hurty [17] and Craig and Bampton [11] for frequency response analysis of complex structures. It is a one-shot projection method which is efficient if only eigenvectors and eigenvalues at the

lower end of the spectrum or close to an excitation frequency [13, 20] are needed with a moderate accuracy. An evaluation of the AMLS method is contained in [15].

We first consider the Component Mode Synthesis (CMS) method, which is an essential ingredient for AMLS. CMS assumes that the degrees of freedom are partitioned into one interface and several sub-structures so that the sub-structures are decoupled in K and M, i.e. for given indices i, j belonging to distinct sub-structures it holds that $K_{ij} = M_{ij} = 0$. This can be done efficiently by graph partitioners like METIS [19] or CHACO [16], based on the sparsity pattern of the matrices.

We denote matrix blocks associated with the interface by 1 and the sub-structures by 2. Reordering rows and columns in (2.1) according to this partition we obtain the pencil

$$\left(\begin{pmatrix} K_{22} & K_{21} \\ K_{21}^T & K_{11} \end{pmatrix}, \begin{pmatrix} M_{22} & M_{21} \\ M_{21}^T & M_{11} \end{pmatrix}\right)$$

which has the same eigenvalues as (2.1) and the same eigenvectors except for permuted components. The next step is to eliminate the off diagonal blocks in K by post- and pre-multiplying this pencil by

$$T = \begin{pmatrix} I & -K_{22}^{-1}K_{21} \\ 0 & I \end{pmatrix}$$

and T^T respectively to obtain the pencil

$$(T^T K T, T^T M T) = \left(\begin{pmatrix} K_{22} & 0 \\ 0 & \tilde{K}_{11} \end{pmatrix}, \begin{pmatrix} M_{22} & \tilde{M}_{21} \\ \tilde{M}_{21}^T & \tilde{M}_{11} \end{pmatrix}\right) \tag{2.2}$$

which again has unchanged eigenvalues and eigenvectors transformed by T. Notice that the decoupling is quite inexpensive since K_{22} is a block diagonal matrix such that the elimination can be performed in parallel.

To reduce the dimension of problem (2.2) we consider the spectral decomposition $K_{22}\Phi = M_{22}\Phi\Lambda$ of the substructure system (which due to its block structure also can be computed inexpensively), and we project problem (2.2) by a block diagonal matrix $\mathrm{diag}(\tilde{\Phi}, I)$ where the rectangular matrix $\tilde{\Phi}$ contains columnwise certain eigenvectors. An efficient rule is to truncate all eigenvectors corresponding to eigenvalues exceeding a prescribed cut-off frequency ω and to collect the remaining ones in $\tilde{\Phi}$. An alternative rule based on a moment-matching principle was suggested in [2, 22].

Assuming that the eigenvectors are normalized with respect to M_{22} we obtain the pencil

$$\left(\begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & \tilde{K}_{11} \end{pmatrix}, \begin{pmatrix} I & \tilde{\Phi}^T\tilde{M}_{21} \\ \tilde{M}_{21}^T\tilde{\Phi} & \tilde{M}_{11} \end{pmatrix}\right)$$

where $\tilde{\Lambda}$ is a diagonal matrix containing the retained eigenvalues on its diagonal.

Subsuming the blockwise Gaussian elimination and the projection onto selected eigenvectors, we use a projection matrix

$$V = T\begin{pmatrix} \tilde{\Phi} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \tilde{\Phi} & -K_{22}^{-1}K_{21} \\ 0 & I \end{pmatrix}$$

(which is never computed explicitly) to obtain the dense projected eigenvalue problem

$$V^T K V y = \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & \tilde{K}_{11} \end{pmatrix} y = \lambda \begin{pmatrix} I & \tilde{\Phi}^T\tilde{M}_{21} \\ \tilde{M}_{21}^T\tilde{\Phi} & \tilde{M}_{11} \end{pmatrix} y = \lambda V^T M V y \tag{2.3}$$

of much smaller dimension. At the lower end of the spectrum (2.3) is known to have good approximation properties.

With increasing size of problem (2.1) CMS suffers some drawbacks. Coarse partitioning leads to huge sub-structures such that the decoupling and modal reduction becomes costly whereas fine partitioning yields a large projected eigenvalue problem (2.3) which is dense and the numerical solution of which therefore is time-consuming.

Component Mode Synthesis can be generalized to a multilevel algorithm which is known as Automated Multi-Level Sub-structuring. Sub-structures are divided into lower-level sub-structures and one interface recursively until all sub-structures have moderate size. Assume that this recursive partitioning leads to sub-structures and interfaces (which we call structures in summary) on p levels of recursion. Then, starting from the lowest level p CMS is applied on each level $i = p, p-1, \dots, 1$ such that in terms of the CMS description above, level i corresponds to the sub-structures and all other levels correspond to the interface.

Notice that AMLS must be implemented unlike the description above to ensure computational efficiency. Firstly it is important to handle structures on the same partitioning level separately to profit from the decoupling. Furthermore structures must be handled in an appropriate order. If all sub-structures which are connected to an interface on the next higher level have already been computed, the interface should be computed as well to avoid the storage of large dense matrices. Implementation details are contained in [13, 15].

3. AMLS reduction for fluid-solid interaction problems. Our AMLS variant for the fluid-solid interaction problem (1.2) is based on the symmetric eigenproblem

$$\left[\begin{pmatrix} 0 & C & K_s & 0 \\ C^T & 0 & 0 & K_f \\ K_s & 0 & 0 & 0 \\ 0 & K_f & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} M_s & 0 & 0 & 0 \\ 0 & M_f & 0 & 0 \\ 0 & 0 & K_s & 0 \\ 0 & 0 & 0 & K_f \end{pmatrix}\right] x = 0 \tag{3.1}$$

whose eigenpairs resemble those from (1.2). If $(\lambda^2, (x_s^T, x_f^T)^T)$ solves problem (1.2) then

$$(\pm\lambda, \begin{pmatrix} \lambda^2 x_s^T & \pm\lambda x_f^T & \pm\lambda x_s^T & x_f^T \end{pmatrix}^T)$$

are solutions of (3.1) unless $\lambda = 0$.

If $\lambda = 0$ is an eigenvalue of (1.2) then the unphysical constant eigenmode leads to a singular mass matrix in (3.1). Problems arising from the singularity of the mass matrix can be overcome by choosing an appropriate sub-structuring.

We have rewritten the unsymmetric eigenvalue problem as a symmetric one of doubled dimension with desired eigenvalues located at neither end of the spectrum. This seems to have several disadvantages such as computational costs and approximation properties. Actually, the standard AMLS algorithm can be modified without much additional computational effort as follows so that the eigenvalue errors can still be bounded.

Algorithm 1 AMLS for fluid-solid interaction eigenvalue problems

```
1: procedure AMLS_CALLSTRUCT(sub-structure i)
2:     for all j in direct descendants(i) do
3:         AMLS_CALLSTRUCT(j)
4:     end for
5:     AMLS_COMPUTESTRUCT(i)
6: end procedure
```

The graph partitioning is again based on the union of the sparsity structures of the matrices K and M. This gives an $s+f$-dimensional partitioning which can be expanded to an $2(s+f)$-dimensional partitioning so that for $i = 1, \dots, s+f$ the ith and $(i+s+f)$th degree of freedom belong to the same sub-structure or interface. We assume in the following that AMLS is performed

on p levels. Then the matrices can be reordered and one obtains the pencil

$$\left(\begin{pmatrix} K_{pp} & \cdots & K_{1p} \\ \vdots & \ddots & \vdots \\ K_{1p}^T & \cdots & K_{11} \end{pmatrix}, \begin{pmatrix} M_{pp} & \cdots & M_{1p} \\ \vdots & \ddots & \vdots \\ M_{1p}^T & \cdots & M_{11} \end{pmatrix} \right), \tag{3.2}$$

where the first diagonal block pencil (K_{pp}, M_{pp}) corresponds to the sub-structure eigenvalue problems on the lowest level and the last block pencil (K_{11}, M_{11}) corresponds to the highest interface eigenvalue problem. Note that any interface still decouples the eigenvalue problem corresponding to the sub-structure degrees of freedom and all of these decoupled eigenvalue problems have the same block structure as (3.1).

We assume in the sequel that the matrices involved have been generated by the Finite-Element Method so that the degenerate eigensolutions show a global behaviour. We see that the diagonal blocks M_{ii} are positive definite as long as the fluid domain has not been partitioned on one single level. This assumption and the special partitioning guarantees that all matrix pencils involved in the AMLS algorithm have a definite mass matrix and a stiffness matrix with symmetric spectrum.

Algorithm 2 AMLS for fluid-solid interaction eigenvalue problems

procedure AMLS_COMPUTESTRUCT(sub-structure i)
  AMLS_BLOCKGAUSS(i)
  Compute partial eigenvalue decomposition

$$K_{ii} \begin{pmatrix} X_s\Lambda_{ii+}^2 & X_s\Lambda_{ii+}^2 \\ X_f\Lambda_{ii+} & -X_f\Lambda_{ii+} \\ X_s\Lambda_{ii+} & -X_s\Lambda_{ii+} \\ X_f & X_f \end{pmatrix} = M_{ii} \begin{pmatrix} X_s\Lambda_{ii+}^2 & X_s\Lambda_{ii+}^2 \\ X_f\Lambda_{ii+} & -X_f\Lambda_{ii+} \\ X_s\Lambda_{ii+} & -X_s\Lambda_{ii+} \\ X_f & X_f \end{pmatrix} \begin{pmatrix} \Lambda_{ii+} & 0 \\ 0 & -\Lambda_{ii+} \end{pmatrix}$$

  Normalize eigenvectors
  for all j in descendants(i) **do**
    $M_{ji11} \leftarrow M_{ji11}X_s\Lambda_{ii+}^2 + M_{ji12}X_f\Lambda_{ii+}$
    $M_{ji12} \leftarrow M_{ji11}X_s\Lambda_{ii+}^2 - M_{ji12}X_f\Lambda_{ii+}$
    $M_{ji21} \leftarrow M_{ji12}$
    $M_{ji22} \leftarrow M_{ji11}$
  end for
  for all j in ancestors(i) **do**
    $M_{ij11} \leftarrow \Lambda_{ii+}^2 X_s^T K_{ij11} + X_f^T K_{ij41}$
    $M_{ij12} \leftarrow \Lambda_{ii+} X_f^T K_{ij22} + \Lambda_{ii+} X_s^T K_{ij23}$
  end for
  $K_{ii11} \leftarrow \begin{pmatrix} \Lambda_{ii+} & 0 \\ 0 & -\Lambda_{ii+} \end{pmatrix}$
  $M_{ii11} \leftarrow \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$
  $K_{ii12} \leftarrow K_{ii13} \leftarrow K_{ii21} \leftarrow K_{ii24} \leftarrow K_{ii31} \leftarrow K_{ii42} \leftarrow 0$
  $M_{ii22} \leftarrow M_{ii33} \leftarrow M_{ii44} \leftarrow 0$
end procedure

The modified AMLS algorithm consists of two steps on each sub-structure i which are basically the same as in the standard AMLS algorithm. The first step is to transform the current approximating pencil by symmetric block Gauss elimination to an equivalent one by eliminating all off-diagonal blocks K_{ij}, $j \neq i$ corresponding to the current sub-structure. Due to the special block structure of K_{ii}, the computational effort is approximately the same as for real matrices of half the size of K_{ii}. The submatrices K_{jk} and M_{jk}, $j, k < i$ corresponding to higher levels preserve the block structure as in (3.1) and they are blockwise dense or zero.

The second step requires to solve the sub-structure eigenvalue problem (K_{ii}, M_{ii}) which is usually parallelized. This problem is known to have a symmetric spectrum because it has (after

reordering) the same block structure as (3.1). As most of the sub-structures involve either fluid or solid degrees of freedom, the coupling matrix vanishes locally and we can halve the size of the eigenproblem in these cases. Since we are interested in eigenpairs at the lower end of the spectrum of the original eigenvalue problem (1.2), i.e. in eigenpairs of the symmetric eigenproblem (3.1) corresponding to small eigenvalues in modulus the current pencil is projected onto the space spanned by all modes with an eigenfrequency which is by modulus smaller than a prescribed cut-off frequency $\omega > 0$. The reduction process then terminates with a pencil of symmetric matrices which has a symmetric spectrum.

Algorithm 3 AMLS for fluid-solid interaction eigenvalue problems

procedure AMLS_BLOCKGAUSS(sub-structure i)
  for all j in descendants(i) **do**
    for all k in ancestors(i) **do**
      $M_{jk11} \leftarrow M_{jk11} - M_{ji11}K_{ii13}^{-1}K_{ik13}$
      $M_{jk12} \leftarrow M_{jk12} - M_{ji12}K_{ii24}^{-1}K_{ik24}$
    end for
  end for
  for all j in ancestors(i) **do**
    $K_{jj12} \leftarrow K_{jj12} - K_{ij13}^T K_{ii13}^{-1}K_{ij12} - K_{ij21}^T K_{ii24}^{-1}K_{ij42} + K_{ij13}^T K_{ii13}^{-1}K_{ii12}K_{ii24}^{-1}K_{ij24}$
    $K_{jj13} \leftarrow K_{jj13} - K_{ij13}^T K_{ii13}^{-1}K_{ij13}$
    $K_{jj24} \leftarrow K_{jj24} - K_{ij24}^T K_{ii24}^{-1}K_{ij24}$
    $M_{jj11} \leftarrow M_{jj11} - M_{ij11}^T K_{ii13}^{-1}K_{ij13} - K_{ij13}^T K_{ii13}^{-1}M_{ij11}^T + K_{ij13}^T K_{ii13}^{-1}M_{jj11}K_{ii13}^{-1}K_{ij13} - K_{ij21}^T K_{ii24}^{-1}K_{ij21} + K_{ij21}^T K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ij13} + K_{ij13}^T K_{ii13}^{-1}K_{ii12}^T K_{ii24}^{-1}K_{ij21} - K_{ij13}^T K_{ii13}^{-1}K_{ii12}^T K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ij13}$
    $M_{jj22} \leftarrow M_{jj22} - M_{ij22}^T K_{ii24}^{-1}K_{ij24} - K_{ij24}^T K_{ii24}^{-1}M_{ij22}^T + K_{ij24}^T K_{ii24}^{-1}M_{jj22}K_{ii24}^{-1}K_{ij24} - K_{ij12}^T K_{ii13}^{-1}K_{ij12} + K_{ij12}^T K_{ii13}^{-1}K_{ii12}K_{ii24}^{-1}K_{ij24} + K_{ij24}^T K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ij12} - K_{ij24}^T K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ii12}K_{ii24}^{-1}K_{ij24}$
    for all k in ancestors(j) **do**
      $K_{jk12} \leftarrow K_{jk12} - K_{ij13}^T K_{ii13}^{-1}K_{ik12} - K_{ij21}^T K_{ii24}^{-1}K_{ik24} + K_{ij13}^T K_{ii13}^{-1}K_{ii12}K_{ii24}^{-1}K_{ik24}$
      $K_{jk21} \leftarrow K_{jk21} - K_{ij24}^T K_{ii24}^{-1}K_{ik21} - K_{ij12}^T K_{ii13}^{-1}K_{ik13} + K_{ij24}^T K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ik13}$
      $K_{jk13} \leftarrow K_{jk13} - K_{ij13}^T K_{ii13}^{-1}K_{ik13}$
      $K_{jk24} \leftarrow K_{jk24} - K_{ij24}^T K_{ii24}^{-1}K_{ik24}$
      $M_{jk11} \leftarrow M_{jk11} - M_{ij11}^T K_{ii13}^{-1}K_{ik13} - K_{ij13}^T K_{ii13}^{-1}M_{ik11}^T + K_{ij13}^T K_{ii13}^{-1}M_{ii11}K_{ii13}^{-1}K_{ik13} - K_{ij21}^T K_{ii24}^{-1}K_{ik12} + K_{ij21}^T K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ik13} + K_{ij13}^T K_{ii13}^{-1}K_{ii12}K_{ii24}^{-1}K_{ik12} - K_{ij13}^T K_{ii13}^{-1}K_{ii12}K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ik13}$
      $M_{jk22} \leftarrow M_{jk22} - M_{ij22}^T K_{ii24}^{-1}K_{ik24} - K_{ij24}^T K_{ii24}^{-1}M_{ik22}^T + K_{ij24}^T K_{ii24}^{-1}M_{ii22}K_{ii24}^{-1}K_{ik24} - K_{ij12}^T K_{ii13}^{-1}K_{ik12} + K_{ij12}^T K_{ii13}^{-1}K_{ii12}K_{ii24}^{-1}K_{ik24} + K_{ij24}^T K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ik12} - K_{ij24}^T K_{ii24}^{-1}K_{ii12}^T K_{ii13}^{-1}K_{ii12}K_{ii24}^{-1}K_{ik24}$
    end for
  end for
  for all j in ancestors(i) **do**
    $M_{ij11} \leftarrow M_{ij11} - M_{ii11}K_{ii13}^{-1}K_{ij13}$
    $M_{ij22} \leftarrow M_{ij22} - M_{ii22}K_{ii24}^{-1}K_{ij24}$
    $M_{ij32} \leftarrow -K_{ij12} + K_{ii12}K_{ii24}^{-1}K_{ij24}$
    $M_{ij41} \leftarrow -K_{ij21}^T + K_{ii12}^T K_{ii13}^{-1}K_{ij13}$
    $K_{ij12} \leftarrow K_{ij13} \leftarrow K_{ij21} \leftarrow K_{ij24} \leftarrow K_{ij31} \leftarrow K_{ij42} \leftarrow 0$
  end for
end procedure

Unlike the representation above, AMLS should be implemented structurewise instead of levelwise to profit from decoupled sub-structures. A precise description is given in Algorithms 1 –

3.

To refer to the relationship between structures we use a tree concept. We call a structure j a direct descandant of structure i if referring to some partitioning level, i is the interface and j a sub-structure. A structure j is called a descandant of i if it can be reached via direct descendants, and it is called an ancestor if i is a descandant of j.

All matrices are subscripted by four numbers (i, j, k, l). The indices (i, j) stand for the matrix block connecting the structures i and j. Each of these matrix blocks is divided by (k, l) into up to 4×4 subblocks such as in (3.1). If structure i as a descendent of structure j has been computed K_{ij} and M_{ij} are merged into 2×4 subblocks matrices. If structure j has been computed as well, K_{ij} and M_{ij} are 2×2 subblocks matrices. If all structures have been computed the assembled projected stiffness and mass matrices are real symmetric and can be stated in the arrowhead-structure known from standard AMLS.

Notice that the Algorithms 1 – 3 resemble standard AMLS in the following sense. If we omit in Algorithm 3 in each of the lines 12, 13, 19, and 20 the last four Schur updates, we construct the same approximating subspace as standard AMLS does for the decoupled problem

$$\begin{pmatrix} K_s & 0 \\ 0 & K_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ 0 & M_f \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix}.$$

Roughly speaking, these updates are responsible for taking into account the coupling in the projection space.

4. An a priori bound. This section is concerned with the error introduced by the truncation of modes. First we consider the version of Component Mode Synthesis (CMS) of the algorithm described above, which is mathematically equivalent to the AMLS method on two levels with a prescribed cut-off frequency on the sub-structures and no eigenvalue truncation on the interface. An a priori CMS-bound for fluid-structure interaction problems can be derived similarly as for standard problems as done in [14]. Afterwards, this bound can be generalized to AMLS by recursive application.

We consider the case that K_f is singular and $\mathrm{Ker}(K_f) = \mathrm{span}\{(1, \ldots, 1)^T\}$ which results from a Finite Element discretization with a nodal basis for the fluid–solid interaction problem. If K_f is positive definite then the presentation becomes even easier and has to be modified in an obvious way.

Starting point for Component Mode Synthesis is a partitioning of the degrees of freedom into interface and sub-structure degrees of freedom. To obtain definite diagonal blocks in the mass matrix we assume throughout this section that the fluid degrees of freedom have not been assigned to one single partitioning level. Reordering the stiffness and mass matrices according to the partitioning the eigenvalue problem can be rewritten equivalently as the pencil

$$\left(\begin{pmatrix} K_{22} & K_{21} \\ K_{21}^T & K_{11} \end{pmatrix}, \begin{pmatrix} M_{22} & M_{21} \\ M_{21}^T & M_{11} \end{pmatrix} \right),$$

where all appearing matrix blocks have the same structure as K and M respectively in (3.1).

The next steps are to eliminate the off-diagonal blocks in K,

$$\left(\begin{pmatrix} K_{22} & 0 \\ 0 & \hat{K}_{11} \end{pmatrix}, \begin{pmatrix} M_{22} & \tilde{M}_{21} \\ \tilde{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \right)$$

to compute the sub-structure eigenvalue decomposition $K_{22}\Phi = M_{22}\Phi\Lambda$ and to transform the pencil by means of the eigenvectors to obtain

$$\left(\begin{pmatrix} \Lambda & 0 \\ 0 & \hat{K}_{11} \end{pmatrix}, \begin{pmatrix} I & \hat{M}_{21} \\ \hat{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \right). \tag{4.1}$$

By Sylvester's law of inertia, we know that the stiffness matrix in (4.1) has a twodimensional kernel and furthermore the same number of positive and negative eigenvalues.

In the following we will distinguish between truncated and retained eigenvalues and between positive eigenvalues plus one zero eigenvalue and their negative counter piece. We assume that the eigenvectors contained in Φ have been reordered so that

$$\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2) = \operatorname{diag}(\Lambda_{1+}, -\Lambda_{1+}, \Lambda_2)$$

where Λ_{1+} collects the positive truncated eigenvalues and Λ_2 the retained ones. This allows a representation of (4.1) as

$$\begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \hat{K}_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} I & 0 & \breve{M}_{21} \\ 0 & I & \check{M}_{21} \\ \breve{M}_{21}^T & \check{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.2}$$

If λ is not a diagonal entry of Λ_1 the problem (4.2) can be transformed by eliminating x_1 into a nonlinear eigenvalue problem of reduced dimension n

$$T(\lambda)x = 0, \tag{4.3}$$

where the eigenparameter λ is restricted to an open interval $J := (-\omega, \omega)$, and

$$T(\lambda) = -\begin{pmatrix} \Lambda_2 & 0 \\ 0 & \tilde{K}_{11} \end{pmatrix} + \lambda \begin{pmatrix} I & \check{M}_{21} \\ \check{M}_{21}^T & \hat{M}_{11} \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T (\Lambda_1 - \lambda I)^{-1} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}. \tag{4.4}$$

Under certain assumptions, eigenvalues of nonlinear eigenvalue problems can be characterized as minmax values of a Rayleigh functional similar to the variational characterization of Poincaré in the linear case. Let the nonlinear eigenvalue problem $T(\lambda)x = 0$ be given with a family of symmetric matrices $T(\lambda)$ defined on a open real interval J. We assume that for every fixed $x \in \mathbb{R}^n \setminus \{0\}$ the real function $f(\lambda, x) := x^T T(\lambda) x$ is continuously differentiable, and that the equation

$$f(\lambda, x) = 0 \tag{4.5}$$

has at most one solution $\lambda \in J$. Then (4.5) implicitly defines a functional p on some subset D of $\mathbb{R}^n \setminus \{0\}$. For a linear problem $T(\lambda) = \lambda M - K$ this is exactly the Rayleigh quotient, and therefore p is called the Rayleigh functional of $T(\lambda)x = 0$.

We further assume that

$$\frac{\partial}{\partial \lambda} f(\lambda, x)\Big|_{\lambda = p(x)} > 0 \quad \text{for all } x \in D \tag{4.6}$$

generalizing the definiteness requirement for M in the linear case. Then the implicit function theorem implies that D is an open set and that p is continuously differentiable on D.

For nonlinear eigenproblems, an appropriate enumeration arises from variational principles for linear eigenvalue problems. If (λ, x) is an eigenpair of the nonlinear eigenvalue problem, then $(0, x)$ is an eigenpair of the matrix $T(\lambda)$ for which a natural enumeration of eigenvalues is given by the max-min characterization in the linear case. Therefore we call an eigenvalue λ of the nonlinear eigenproblem a k-th eigenvalue if zero is the kth largest eigenvalue of the matrix $T(\lambda)$.

Using this enumeration, the following variational characterization for nonoverdamped systems (i.e. $D \neq \mathbb{R}^n \setminus \{0\}$) has been proved by Voss and Werner in [29].

PROPOSITION 4.1. *Under the conditions given above the following conclusion hold:*

i) For every $k \in \mathbb{N}$ there is at most one kth eigenvalue of problem (4.3) which can be characterized by

$$\lambda_k = \min_{dim\ V = k} \sup_{0 \neq x \in V \cap D} p(x) \tag{4.7}$$

The minimum is attained by the invariant subspace corresponding to the kth largest eigenvalues of $T(\lambda_k)$ and the supremum is attained by all eigenvectors of (4.3) corresponding to λ_k

ii) If

$$\lambda_k = \inf_{dim\ V=k} \sup_{0\neq x\in V\cap D} p(x) \in J$$

for some $k \in \mathbb{N}$ then λ_k is the kth eigenvalue of (4.3) and (4.7) *holds.*

The nonlinear eigenvalue problem (4.3) satisfies the assumptions of proposition 4.1. For all $\lambda \in J$ and $x = \begin{pmatrix} x_2^T & x_3^T \end{pmatrix}^T \in D$ it holds that

$$\begin{aligned}
\frac{\partial}{\partial\lambda} f(\lambda, x) =& x^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} x + x^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T (\Lambda_1 - \lambda I)^{-2} (2\lambda\Lambda_1 - \lambda^2 I) \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} x \\
=& x^T \left[\begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \breve{M}_{21}^T \breve{M}_{21} \end{pmatrix} \right] x \\
&+ x^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T \left[(\Lambda_1 - \lambda I)^{-2} (2\lambda\Lambda_1 - \lambda^2 I) + I \right] \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} x \\
=& x^T \left[\begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \breve{M}_{21}^T \breve{M}_{21} \end{pmatrix} \right] x \\
&+ x^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T \begin{pmatrix} D_1 + I & 0 \\ 0 & D_2 + I \end{pmatrix} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} x,
\end{aligned}$$

where $D_1 = (\Lambda_{1+} - \lambda I)^{-2}(2\lambda\Lambda_{1+} - \lambda^2 I)$ and $D_2 = (-\Lambda_{1+} - \lambda I)^{-2}(-2\lambda\Lambda_{1+} - \lambda^2 I)$. The matrix in the first quadratic form is the Schur complement of the unreduced mass matrix and therefore positive semi-definite. The matrices $D_1 + I$ and $D_2 + I$ in the second quadratic form are positive definite, too, and thus

$$\frac{\partial}{\partial\lambda} f(\lambda, x) \geq 0.$$

The first quadratic form is positive unless $\begin{pmatrix} x_1^T & x_2^T & x_3^T \end{pmatrix}^T$ is contained in the kernel of the unreduced mass matrix. In this case, $\begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} x = -x_1$ and the second quadratic form is positive. In both cases,

$$\frac{\partial}{\partial\lambda} f(\lambda, x) > 0.$$

Beside the enumeration in Proposition 4.1 we will introduce a different numbering which is convenient for (4.4). We denote by $0 = \lambda_{+1} = \lambda_{-1}$ the zero eigenvalues of (4.3) and by

$$-\omega < \cdots \leq \lambda_{-3} \leq \lambda_{-2} < 0 < \lambda_{+2} \leq \lambda_{+3} \leq \cdots < \omega$$

the remaining eigenvalues in J. Then it holds that $\lambda_{-j} = -\lambda_{+j}$.

The matrix $T(0)$ has rank $n - 2$ and a symmetric spectrum, i.e. the two zero eigenvalues have the enumeration $n/2$ and $n/2 + 1$ in terms of the theory of nonlinear eigenvalue problems. Proposition 4.1 yields

$$\lambda_{+j} = \min_{\dim\ V=n/2+j} \sup_{x\in V\cap D} p(x) \quad \text{and} \quad \lambda_{-j} = \min_{\dim\ V=n/2-j+1} \sup_{x\in V\cap D} p(x).$$

The CMS a priori bounds in Proposition 4.2 can be derived by comparing the nonlinear Rayleigh functional p with the Rayleigh quotient r of the truncated linear eigenproblem

$$\begin{pmatrix} \Lambda_2 & 0 \\ 0 & \hat{K}_{11} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \tag{4.8}$$

on appropriate subspaces.

PROPOSITION 4.2. *Denote by $0 = \lambda_{+1} < \lambda_{+2} \leq \lambda_{+3} \leq \cdots < \omega$ one zero eigenvalue and the positive eigenvalues of the nonlinear eigenvalue problem and by $0 = \tilde{\lambda}_{+1} < \tilde{\lambda}_{+2} \leq \tilde{\lambda}_{+3} \leq \cdots < \omega$ the corresponding eigenvalues of the truncated linear eigenproblem* (4.8).

Then for every $j \geq 2$ such that $\lambda_{+j}, \tilde{\lambda}_{+j} \in J := (-\omega, \omega)$ it holds that

$$\lambda_{+j} - \frac{\lambda_{+j}^2}{\omega + \lambda_{+j}} \leq \tilde{\lambda}_{+j} \leq \lambda_{+j} + \frac{\lambda_{+j}^2}{\omega - \lambda_{+j}}, \tag{4.9}$$

i.e.

$$-\frac{\lambda_{+j}}{\omega + \lambda_{+j}} \leq \frac{\tilde{\lambda}_{+j} - \lambda_{+j}}{\lambda_{+j}} \leq \frac{\lambda_{+j}}{\omega - \lambda_{+j}}. \tag{4.10}$$

REMARK 4.1. *The proof is given for the case that K_f is singular. For positive definite K_f the bounds* (4.10) *also hold for $j = 1$.*

Proof. To prove the upper bound, consider the $(n/2 + j)$-dimensional subspace W so that

$$\lambda_{+j} = \sup_{x \in W \cap D} p(x).$$

Denote by $s(x)$ the Rayleigh quotient corresponding to the matrix $\lambda_{+j} I - T(\lambda_{+j})$, i.e.

$$\lambda_{+j} = \min_{\dim V = n/2+j} \max_{0 \neq x \in V} s(x) = \max_{0 \neq x \in W} s(x).$$

Then $s(x) \leq \lambda_{+j}$ for all nonzero $x \in W$, and therefore

$$-x^T \begin{pmatrix} \Lambda_2 & 0 \\ 0 & \tilde{K}_{11} \end{pmatrix} x + \lambda_{+j} x^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} x + \lambda_{+j}^2 x^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T (\Lambda_1 - \lambda_{+j} I)^{-1} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} x \geq 0.$$

Hence

$$\lambda_{+j} \geq \frac{x^T \begin{pmatrix} \Lambda_2 & 0 \\ 0 & \tilde{K}_{11} \end{pmatrix} x}{x^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} x} - \lambda_{+j}^2 \frac{x^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T (\Lambda_1 - \lambda_{+j} I)^{-1} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} x}{x^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} x}.$$

This holds in particular for $\hat{x} \in W$ which maximizes the Rayleigh quotient of the truncated eigenproblem, i.e. $r(\hat{x}) = \max_{0 \neq x \in W} r(x)$, from which we obtain

$$\begin{aligned}
\lambda_{+j} &\geq \max_{0 \neq x \in W} r(x) - \lambda_{+j}^2 \frac{\hat{x}^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T (\Lambda_1 - \lambda_{+j} I)^{-1} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} \hat{x}}{\hat{x}^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \hat{x}} \\
&\geq \min_{\dim V = n/2+j} \max_{0 \neq x \in V} r(x) - \lambda_{+j}^2 \frac{\hat{x}^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T (\Lambda_1 - \lambda_{+j} I)^{-1} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} \hat{x}}{\hat{x}^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \hat{x}} \\
&= \tilde{\lambda}_{+j} - \lambda_{+j}^2 \frac{\hat{x}^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T (\Lambda_1 - \lambda_{+j} I)^{-1} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} \hat{x}}{\hat{x}^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \hat{x}}.
\end{aligned}$$

The foregoing fraction can be bounded by

$$\frac{\hat{x}^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T (\Lambda_1 - \lambda_{+j} I)^{-1} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} \hat{x}}{\hat{x}^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \hat{x}} \leq \frac{1}{\omega - \lambda_{+j}} \frac{\hat{x}^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} \hat{x}}{\hat{x}^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \hat{x}},$$

because it follows by comparison of the diagonal elements that the matrix

$$\frac{1}{\omega - \lambda} I - (\Lambda_1 - \lambda I)^{-1}$$

is positive definite for $\lambda \in J$. Hence

$$\lambda_{+j} \geq \tilde{\lambda}_{+j} - \frac{\lambda_{+j}^2}{\omega - \lambda_{+j}} \frac{\hat{x}^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}^T \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} \hat{x}}{\hat{x}^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} \hat{x}}.$$

From the positive semi-definiteness of the transformed mass matrix it follows that its Schur complement

$$\begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \hat{M}_{11} \end{pmatrix} - \begin{pmatrix} 0 \\ \breve{M}_{21}^T \end{pmatrix} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix}$$

is positive semi-definite as well. Thus

$$\max_{x \in W} \frac{x^T \begin{pmatrix} 0 \\ \breve{M}_{21}^T \end{pmatrix} \begin{pmatrix} 0 & \breve{M}_{21} \end{pmatrix} x}{x^T \begin{pmatrix} I & \breve{M}_{21} \\ \breve{M}_{21}^T & \tilde{M}_{11} \end{pmatrix} x} \leq 1,$$

from which we get

$$\lambda_{+j} \geq \tilde{\lambda}_{+j} - \frac{\lambda_{+j}^2}{\omega - \lambda_{+j}}.$$

To prove the lower bound, one can derive an upper bound

$$\tilde{\lambda}_{-j} \leq \lambda_{-j} + \frac{\lambda_{-j}^2}{\omega - \lambda_{-j}}$$

for the corresponding negative eigenvalues by the same technique as above with the substitution $n/2 + j \leftrightarrow n/2 - j + 1$, $\lambda_{-j} \leftrightarrow \lambda_{+j}$ and $\tilde{\lambda}_{-j} \leftrightarrow \tilde{\lambda}_{+j}$. The lower bound for the positive eigenvalues follows by the symmetry of the spectrum. □

The upper bound in (4.10) to the relative error has the same structure as the error bound given in [14] for CMS applied to definite eigenvalue problem $Kx = \lambda Mx$. In the definite case the lower bound is 0 due to the fact that CMS is a projection method and the eigenvalues under consideration are at the lower end of the spectrum. CMS as introduced in Section 3 for the indefinite eigenproblem (3.1) discards eigenmodes corresponding to eigenvalues which exceed a cut-off frequency in modulus. Since we are approximating an interior eigenvalue λ_{+j} of the nonlinear eigenvalue problem (4.4) by an interior eigenvalue $\tilde{\lambda}_{+j}$ of the linear eigenvalue problem (4.6) which may have different numbers with respect to the natural enumeration (smallest eigenvalue first one, second smallest second one, etc.) it may happen that $\tilde{\lambda}_{+j} < \lambda_{+j}$. The following example demonstrates that this may actually happen.

We apply CMS to the fluid-solid type eigenvalue problem

$$\begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} x = \lambda \begin{pmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -2 & -2 & 5 & 1 \\ -2 & -2 & 1 & 5 \end{pmatrix} x \tag{4.11}$$

as described above. Its eigenvalues in the symmetrized form are approximately given by ± 0.5096, ± 0.5776, ± 0.7071, and ± 1.2411. We set $\omega = 0.75$ and take the first and third degree of freedom of (4.11) as substrucutre nodes and the remaining two as interface nodes. Then one eigenvalue pair is truncated and the eigenvalues of the resulting 6×6 pencil are given by ± 0.5069, ± 0.5885, and ± 0.7620, i.e. the smallest positive eigenvalue has been decreased in the reduction process.

The CMS bounds in Proposition 4.2 is sharp. Assume that we apply fluid-solid interaction CMS on the parameter dependant example (1.5) using again the cut-off frequency $\omega = \sqrt{1.5}$.

Given a partitioning so that the solid degree of freedom is an interior degree and the fluid degree of freedom is an interface degree we can write the symmetric formulation as the pencil

$$\left(\begin{pmatrix} 0 & 2 & a & 0 \\ 2 & 0 & 0 & 0 \\ a & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right),$$

where the rows and columns have been reordered according to the partitioning, i.e. the upper left 2×2 block corresponds to the interior degree and the lower right 2×2 block to the interface degree. Under these premises, one eigenpair is truncated independently of a.

If $a \to \infty$, the exact smallest non-negative eigenvalue λ_{+1} and its approximation $\tilde{\lambda}_{+1}$ tend to zero. Comparing the upper bound $\lambda_{+1} + \lambda_{+1}^2/(\omega - \lambda_{+1})$ with the approximated eigenvalue $\tilde{\lambda}_{+1}$ relative to the exact eigenvalue λ_{+1}, it holds that

$$0 \leq \frac{\lambda_{+1} + \frac{\lambda_{+1}^2}{\omega - \lambda_{+1}} - \tilde{\lambda}_{+1}}{\lambda_{+1}} = \frac{\lambda_{+1} - \tilde{\lambda}_{+1}}{\lambda_{+1}} + \frac{\lambda_{+1}}{\omega - \lambda_{+1}} \leq \frac{\lambda_{+1}}{\omega + \lambda_{+1}} + \frac{\lambda_{+1}}{\omega - \lambda_{+1}},$$

where we used the lower bound to $\tilde{\lambda}_{+1}$. For $a \to \infty$, both summands on the right tend to zero, and it follows that

$$0 \leq \frac{\lambda_{+1} + \frac{\lambda_{+1}^2}{\omega - \lambda_{+1}} - \tilde{\lambda}_{+1}}{\lambda_{+1}} \to 0 \quad \text{for } a \to \infty.$$

Similarly, one obtains

$$0 \leq \frac{\tilde{\lambda}_{+1} - \lambda_{+1} + \frac{\lambda_{+1}^2}{\omega + \lambda_{+1}}}{\lambda_{+1}} = \frac{\tilde{\lambda}_{+1} - \lambda_{+1}}{\lambda_{+1}} + \frac{\lambda_{+1}}{\omega + \lambda_{+1}} \leq \frac{\lambda_{+1}}{\omega - \lambda_{+1}} + \frac{\lambda_{+1}}{\omega + \lambda_{+1}},$$

where we used the upper bound to $\tilde{\lambda}_{+1}$. It follows that

$$0 \leq \frac{\tilde{\lambda}_{+1} - \lambda_{+1} + \frac{\lambda_{+1}^2}{\omega + \lambda_{+1}}}{\lambda_{+1}} \to 0 \quad \text{for } a \to \infty,$$

and therefore both eigenvalue bounds are asymptotically sharp.

AMLS on p partitioning levels is mathematically equivalent to p CMS steps, so that in CMS step on level $k = p, \ldots, 1$ eigenmodes on level k are truncated and eigenmodes on all other levels are retained. We denote by $\lambda_{+j}^{(k)}$ the eigenvalue approximation if the lowest k partitioning levels have been handled, i.e. $\lambda_{+j}^{(0)}$ denotes the exact eigenvalues and $\lambda_{+j}^{(p)}$ the approximation when the reduction process has terminated. Then we apply the CMS bound in proposition 4.2 recursively and obtain the following error bound for AMLS.

PROPOSITION 4.3. *Consider the AMLS algorithm for fluid-solid interaction eigenproblems on p levels. Denote by $\lambda_{+j}^{(k)}$ the eigenvalues after the k lowest partitioning levels have been handled ($k = 0, \ldots, p$) and assume that the cut-off frequency satisfies $\omega > p\lambda_{+j}^{(0)} \geq 0$. Then the eigenvalues can be bounded by*

$$\frac{\omega \lambda_{+j}^{(p)}}{\omega + p\lambda_{+j}^{(p)}} \leq \lambda_{+j}^{(0)} \leq \frac{\omega \lambda_{+j}^{(p)}}{\omega - p\lambda_{+j}^{(p)}} \tag{4.12}$$

Proof. Proposition 4.2 yields for $k = 0, \ldots, p-1$

$$\lambda_{+j}^{(k)} \leq \frac{\omega \lambda_{+j}^{(k+1)}}{\omega - \lambda_{+j}^{(k+1)}} \quad \text{and} \quad \lambda_{+j}^{(k+1)} \leq \frac{\omega \lambda_{+j}^{(k)}}{\omega - \lambda_{+j}^{(k)}}.$$

Then it holds that

$$\frac{\omega\lambda_{+j}^{(k)}}{w-k\lambda_{+j}^{(k)}} \le \frac{\omega\lambda_{+j}^{(k+1)}}{\omega-(k+1)\lambda_{+j}^{(k+1)}} \quad \text{and} \quad \frac{\omega\lambda_{+j}^{(k)}}{w+(k-p)\lambda_{+j}^{(k)}} \le \frac{\omega\lambda_{+j}^{(k-1)}}{\omega+(k-p-1)\lambda_{+j}^{(k-1)}}$$

as long as $\omega > p\lambda^{(k)}$. We obtain by repeated application the eigenvalue bounds

$$\lambda_{+j}^{(0)} \le \frac{\omega\lambda_{+j}^{(1)}}{\omega-\lambda_{+j}^{(1)}} \le \frac{\omega\lambda_{+j}^{(2)}}{\omega-2\lambda_{+j}^{(2)}} \le \ldots \le \frac{\omega\lambda_{+j}^{(p)}}{\omega-p\lambda_{+j}^{(p)}} \quad \text{and}$$
$$\lambda_{+j}^{(p)} \le \frac{\omega\lambda_{+j}^{(p-1)}}{\omega-\lambda_{+j}^{(p-1)}} \le \frac{\omega\lambda_{+j}^{(p-2)}}{\omega-2\lambda_{+j}^{(p-2)}} \le \ldots \le \frac{\omega\lambda_{+j}^{(0)}}{\omega-p\lambda_{+j}^{(0)}}$$

for the multi-level algorithm. □

5. Numerical results. To evaluate the modified AMLS algorithm for fluid-solid interaction problems, we consider a two-dimensional model with 120473 degrees of freedom. The solid is steel and its discretization has 67616 degrees of freedom. As fluid we consider water whose discretization lead to 52857 degrees of freedom. To investigate the coupling effects, the underlying geometry

No.	coupled	uncoupled	No.	coupled	uncoupled
1	0.0	0.0	11	561.2	472.4
2	28.0	56.8	12	650.4	562.2
3	41.5	75.5	13	746.2	683.3
4	92.7	151.0	14	810.1	683.9
5	124.0	178.6	15	860.9	892.2
6	138.2	186.6	16	992.5	963.6
7	270.3	210.5	17	1009.5	986.3
8	321.7	225.5	18	1146.9	1103.9
9	388.7	402.9	19	1216.8	1147.5
10	402.7	451.7	20	1321.8	1339.2

TABLE 5.1
First natural frequencies [Hz]

was chosen with a rather large interface between fluid and solid. In table 5.1 we compare the first natural frequencies of the coupled model with the results obtained by neglecting the coupling effects.

We applied the AMLS variant described in Section 3 to the coupled fluid-solid problem and compared the eigenvalue approximations to those obtained from the standard procedure touched on in the introduction and to the a priori error bound from Section 4. In both cases, the algorithm was performed on 10 sub-structuring levels and 751 structures using a cut-off frequency corresponding to 10000 Hz on each partitioning level.

Eigenvalues with large accuracy improvements (e.g. $\lambda \approx 100$ Hz) turned out to belong to eigenforms with significant influence of the coupling. Eigenforms corresponding to larger eigenfrequencies were less influenced by the coupling and in some cases, the eigenvalue approximations are slightly worse compared with the AMLS variant neglecting the coupling effects in the reduction process. In all cases, the eigenvalue approximations were of larger magnitude than the exact eigenvalues.

As in standard Automated Multi-Level Sub-structuring for symmetric eigenproblems, the eigenvalue bound highly overestimates the relative error. Such an error bahaviour is typical for applications with a reasonable partitioning of the degrees of freedom. Nevertheless, malicious examples can be constructed which exploit the a priori bound entirely, see the remark in Section 4.

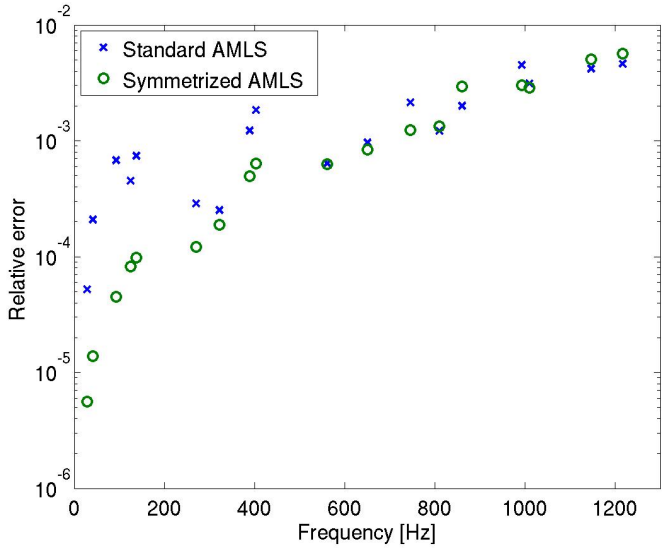

FIG. 5.1. *Relative error of adapted AMLS for fluid-solid interaction problems and relative error of standard AMLS applied to fluid-solid interaction problems*

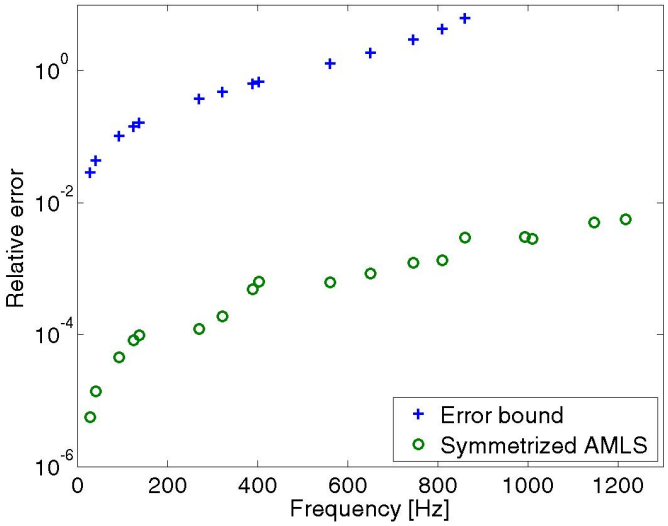

FIG. 5.2. *Relative error of adapted AMLS for fluid-solid interaction problems compared with the relative error obtained by the a priori error bound*

6. Concluding remarks. Recent studies in vibro-acoustic analysis for huge symmetric eigenvalue problems where a large number of eigenpairs with moderate accuracy are needed have shown that for this type of problems AMLS is considerably faster than Lanczos type approaches. Unsymmetric eigenproblems governing free vibrations of fluid-solid structures are covered in praxis by first neglecting the coupling terms, determining a large number of approximate eigenmodes of the uncoupled system by AMLS, and projecting the original problem to the space spanned by these modes. Based on a symmetrization of doubled dimension we developed a variant of the Automated Multi-Level Sub-structuring method for fluid-solid system which requires essentially the same cost as AMLS for a symmetric problem of the same dimension as the unsymmetric problem. For strongly coupled problem this improves the approximation properties substantially. An

a priori error bound is proved which usually overestimates the errors by orders of magnitude but which can not be improved without further assumptions.

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