NONLINEAR LOW RANK MODIFICATION OF A SYMMETRIC EIGENVALUE PROBLEM

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Abstract. This paper studies existence and uniqueness results and interlacing properties of nonlinear modifications of small rank of symmetric eigenvalue problems. Approximation properties of the Rayleigh functional are used to design numerical methods the local convergence of which is quadratic or even cubic. Numerical examples demonstrate their efficiency.

Key words. eigenvalue problem, nonlinear small rank modification, interlacing property

AMS subject classifications. 15A18, 47J10, 65F15

1. Introduction.

We consider a nonlinear low rank modification of a symmetric eigenvalue problem

$$(A + \phi(\lambda)H)\, x = \lambda x \tag{1.1}$$

where $A, H \in \mathbb{C}^{n\times n}$ are Hermitian matrices, H has a low rank $k << n$, and ϕ is real valued and continuous. Problem (1.1) generalizes the constant rank-one modification

$$(A + \tau cc^H)x = \lambda x \tag{1.2}$$

with $c \in \mathbb{C}^n$ and $\tau \in \mathbb{R}$ or small rank modifications $(A + H)x = \lambda x$ of a symmetric matrix A [2, 8]. Nonlinear modifications of this type emerge for $k = 1$ from the study of free vibrations of mechanical structures with an attached load [10] or in fiber optics modelling [5, 6, 7] and for $k > 1$ they govern free vibrations of fluid-solid structures [1, 11].

In a recent paper Huang, Bai and Su [4] studied nonlinear rank-one modifications of symmetric eigenvalue problems. Under the conditions that ϕ is of one sign and $\phi'(\lambda) \le 0$ they proved the existence of eigenvalues of problem (1.1) and a uniqueness result, and interlacing properties between eigenvalues of (1.1) and the matrix A, and they presented three numerical methods. In this paper we generalize these results in several respects: we relax the requirements for the function ϕ, we prove corresponding results for $k > 1$, and we present a method which converges with cubic order of convergence.

Our paper is organized as follows. In Section 2 we prove the existence and uniqueness of eigenvalues of a rank-one modification of a symmetric eigenvalue problem, where we require only $\phi'(\lambda)\|c\|^2 < 1$ close to the eigenvalue under consideration. If this condition is satisfied globally the eigenvalues of (1.2) interlace the ones of A. Section 3 generalizes these results to low rank modifications (1.1), and in Section 4 we propose a numerical method for both types of problems, the local convergence of which is quadratic or even cubical. Termination of these methods is based on an error bound which comes for free. The paper concludes with numerical examples in Section 5 demonstrating the efficiency of the methods.

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2. Rank one perturbation. In the following $A \in \mathbb{R}^{n\times n}$ always denotes a symmetric matrix. Let $\alpha_1 \le \ldots \le \alpha_n$ be the eigenvalues of A, and let $\alpha_0 := -\infty$ and $\alpha_{n+1} = \infty$.

The following interlacing theorem for constant rank-one modifications of Hermitian matrices is well known (cf. [3]).

THEOREM 2.1. *Let $B := A + \tau cc^T$, $c \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$, with eigenvalues $\beta_1 \le \ldots \le \beta_n$. Then it holds that*

$$\alpha_i \le \beta_i \le \alpha_{i+1} \quad \text{for } \tau > 0,\ i = 1, \ldots, n \tag{2.1}$$
$$\alpha_{i-1} \le \beta_i \le \alpha_i \quad \text{for } \tau < 0,\ i = 1, \ldots, n. \tag{2.2}$$

If A is diagonal with distinct diagonal entries $\alpha_1 < \ldots < \alpha_n$, and if all components of c are different from zero then (2.1) *and* (2.2) *even hold with strict inequalities.*

From Theorem 2.1 we immediately obtain the following existence result for the nonlinear rank-one modification $B := A + \phi(\lambda)cc^T$ of A.

THEOREM 2.2.

(i) For $k \in \{1, \ldots, n\}$ let $\phi \in C[\alpha_k, \alpha_{k+1}]$ be nonnegative. Then the nonlinear eigenvalue problem

$$(A + \phi(\lambda)cc^T)x = \lambda x \tag{2.3}$$

has an eigenvalue $\hat\lambda \in [\alpha_k, \alpha_{k+1}]$.

(ii) For $k \in \{1, \ldots, n\}$ let $\phi \in C[\alpha_{k-1}, \alpha_k]$ be nonpositive. Then the nonlinear eigenvalue problem 2.3 has an eigenvalue $\hat\lambda \in [\alpha_{k-1}, \alpha_k]$.

Proof. If $\alpha_k = \alpha_{k+1}$ then there exists a vector x in the corresponding eigenspace of A such that $x^T c = 0$, and x is an eigenvector of (2.3) corresponding to $\hat\lambda = \alpha_k$.

Let $\alpha_k < \alpha_{k+1}$ and $\phi(\lambda) \ge 0$ in $[\alpha_k, \alpha_{k+1}]$. For $\lambda \in [\alpha_k, \alpha_{k+1}]$ it follows from Theorem 2.1 that the kth smallest eigenvalue $\mu_k(\lambda)$ of

$$(A + \phi(\lambda)cc^T)x = \mu x \tag{2.4}$$

satisfies $\mu_k(\lambda) \in [\alpha_k, \alpha_{k+1}]$. Hence, $\lambda \mapsto \mu_k(\lambda)$ maps the closed interval $[\alpha_k, \alpha_{k+1}]$ continuously into itself, and therefore has a fixed point $\hat\lambda \in [\alpha_k, \alpha_{k+1}]$, which proves statement (i).

(ii) follows in the same way using (2.2). ◻

THEOREM 2.3. *Assume that the conditions of part (i) of Theorem 2.2 hold and that for some $\delta > 0$ the condition*

$$\frac{\phi(\lambda) - \phi(\mu)}{\lambda - \mu}\|c\|^2 \le 1 - \delta \tag{2.5}$$

is satisfied for $\lambda, \mu \in I := [\alpha_k, \alpha_{k+1}]$, $\lambda \ne \mu$. Then problem (2.3) *has at most one eigenvalue $\hat\lambda \in (\alpha_k, \alpha_{k+1})$.*

Proof. For $\alpha_k = \alpha_{k+1}$ nothing has to be proved. Let $\alpha_k < \alpha_{k+1}$ and $\phi(\lambda) \ge 0$ for $\lambda \in [\alpha_k, \alpha_{k+1}]$.

Let $\lambda \in (\alpha_k, \alpha_{k+1})$ and let $\mu_{k+1}(\lambda)$ be the $(k+1)$th smallest eigenvalue of (2.4). If $\tilde V$ denotes the invariant subspace of $A + \phi(\lambda)cc^T$ corresponding to the $(k+1)$th smallest eigenvalues, then it holds that

$$\begin{aligned}\mu_{k+1}(\lambda) &= \max_{x\in\tilde V} \frac{x^TAx + \phi(\lambda)(c^Tx)^2}{\|x\|^2} \\ &\ge \max_{x\in\tilde V} \frac{x^TAx}{\|x\|^2} \ge \min_{\dim V = k+1} \max_{x\in V} \frac{x^TAx}{\|x\|^2} = \alpha_{k+1},\end{aligned}$$

and if $\mu_{k-1}(\lambda)$ is the $(k-1)$th smallest eigenvalue of (2.4) and $\tilde{v}^j$ is an eigenvector corresponding the jth smallest eigenvalue, then it holds that

$$\begin{aligned}\mu_{k-1}(\lambda) &= \min_{x \in \{\tilde{v}^1,\dots,\tilde{v}^{k-2}\}^\perp} \frac{x^TAx + \phi(\lambda)(c^Tx)^2}{\|x\|^2} \\ &\le \min_{x \in \{\tilde{v}^1,\dots,\tilde{v}^{k-2},c\}^\perp} \frac{x^TAx + \phi(\lambda)(c^Tx)^2}{\|x\|^2} \\ &= \min_{x \in \{\tilde{v}^1,\dots,\tilde{v}^{k-2},c\}^\perp} \frac{x^TAx}{\|x\|^2} \le \max_{\dim V \le k-1} \min_{x \in V^\perp} \frac{x^TAx}{\|x\|^2} = \alpha_k.\end{aligned}$$

Hence, an eigenvalue $\hat{\lambda} \in (\alpha_k, \alpha_{k+1})$ of (2.3) is a fixed point of the mapping $\lambda \mapsto \mu_k(\lambda)$. We prove that under the condition (2.5) for every fixed point $\hat{\lambda} \in (\alpha_k, \alpha_{k+1})$ it holds that

$$\mu_k(\lambda) \left\{ \begin{array}{c} > \\ < \end{array} \right\} \lambda \quad \text{if } \lambda \left\{ \begin{array}{c} < \\ > \end{array} \right\} \hat{\lambda}. \tag{2.6}$$

Then it is obvious that there is at most one fixed point in (α_k, α_{k+1}).

Let $x(\lambda)$ be an eigenvector of (2.4) corresponding to $\mu_k(\lambda)$ with $\|x(\lambda)\| = 1$. Let $\hat{\lambda} = \mu_k(\hat{\lambda}) \in (\alpha_k, \alpha_{k+1})$ be an eigenvalue of (2.3), and assume that $\lambda \neq \hat{\lambda}$. Multiplying (2.4) for $\hat{\lambda}$ from the left by $x(\lambda)^T$, multiplying (2.3) for λ from the left by $x(\hat{\lambda})^T$, and subtracting yields

$$(\phi(\lambda) - \phi(\hat{\lambda}))x(\lambda)^Tcx(\hat{\lambda})^Tc = (\mu_k(\lambda) - \hat{\lambda})x(\lambda)^Tx(\hat{\lambda}).$$

For λ close to $\hat{\lambda}$ we may assume that $x(\lambda)^Tx(\hat{\lambda}) > 0$, and it follows

$$\frac{\mu_k(\lambda) - \hat{\lambda}}{\lambda - \hat{\lambda}} = \frac{\phi(\lambda) - \phi(\hat{\lambda})}{\lambda - \hat{\lambda}} \frac{x(\lambda)^Tcx(\hat{\lambda})^Tc}{x(\lambda)^Tx(\hat{\lambda})}. \tag{2.7}$$

For $(\phi(\lambda) - \phi(\hat{\lambda}))/(\lambda - \hat{\lambda}) > 0$ we further obtain from (2.5) and the continuous dependence of $x(\lambda)$ on λ

$$\begin{aligned}\frac{\mu_k(\lambda) - \hat{\lambda}}{\lambda - \hat{\lambda}} &\le \frac{\phi(\lambda) - \phi(\hat{\lambda})}{\lambda - \hat{\lambda}} \|c\|^2 \frac{\|x(\lambda)\| \cdot \|x(\hat{\lambda})\|}{x(\lambda)^Tx(\hat{\lambda})} \\ &\le (1-\delta) \frac{\|x(\lambda)\| \cdot \|x(\hat{\lambda})\|}{x(\lambda)^Tx(\hat{\lambda})} < 1\end{aligned}$$

if λ is sufficiently close to $\hat{\lambda}$. Hence, for $(\phi(\lambda) - \phi(\hat{\lambda}))/(\lambda - \hat{\lambda}) > 0$ (2.6) holds and for $(\phi(\lambda) - \phi(\hat{\lambda}))/(\lambda - \hat{\lambda}) \le 0$ these inequalities are trivial. □

With an analogous proof we obtain also a uniqueness result for a nonpositive function ϕ.

THEOREM 2.4. *Assume that the conditions of part (ii) of Theorem 2.2 hold and that for some* $\delta > 0$ *the condition*

$$\frac{\phi(\lambda) - \phi(\mu)}{\lambda - \mu} \|c\|^2 \le 1 - \delta \tag{2.8}$$

is satisfied in $I := [\alpha_{k-1}, \alpha_k]$. *Then problem* (2.3) *has at most one eigenvalue* $\hat{\lambda} \in (\alpha_{k-1}, \alpha_k)$.

Some remarks are in order:
REMARK 2.5.

(i) Theorems 2.2, 2.3, and 2.4 also hold for a Hermitian matrix A, a complex vector $c \in \mathbb{C}^n$, and a real valued function ϕ.

(ii) Theorem 2.2 also holds for a rank one modification of a generalized eigenvalue problem

$$(K + \phi(\lambda)cc^H)x = \lambda Mx \tag{2.9}$$

where $K, M \in \mathbb{C}^{n\times n}$ are Hermitian, and M is positive definite. If $M = CC^H$ is the Cholesky factorization of M, (2.9) *is equivalent to*

$$(C^{-1}KC^{-H} + \phi(\lambda)(C^{-1}c)(C^{-1}c)^H)y = \lambda y, \quad y = C^H x.$$

Hence, condition (2.5) *in Theorems 2.3 and 2.4 has to be replaced with*

$$\frac{\phi(\lambda) - \phi(\mu)}{\lambda - \mu} c^H M^{-1} c \le 1 - \delta. \tag{2.10}$$

(iii) In the following we consider only the real, symmetric case and the eigenvalue problem (2.3)*.*

(iv) If ϕ is differentiable in (α_k, α_{k+1}) condition (2.5) *is equivalent to*

$$\phi'(\lambda)\|c\|^2 \le 1 - \delta \quad \text{for every } \lambda \in (\alpha_k, \alpha_{k+1}).$$

(v) Huang, Bai, and Su [4] proved the uniqueness Theorems 2.3 and 2.4 under the more restrictive condition $\phi'(\lambda) \le 0$ for every $\lambda \in (\alpha_k, \alpha_{k+1})$ and $\lambda \in (\alpha_{k-1}, \alpha_k)$, respectively.

In the proof of Theorem 2.3 we discussed the behavior of the function $\lambda \mapsto \mu_k(\lambda)$ in the vicinity of a fixed point $\hat{\lambda}$ under the condition (2.5). Obviously this discussion is independent of the number k of the eigenvalue and the particular interval (α_k, α_{k+1}). Hence, if $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function of one sign such that (2.5) holds for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \ne \mu$, then for every $k \in \{1, \dots, n\}$ the function $\lambda \mapsto \mu_k(\lambda)$ has at most one fixed point, i.e. there is at most one eigenvalue λ_k of the nonlinear eigenvalue problem (2.3) which is the kth smallest eigenvalue of the linear eigenvalue problem $(A + \phi(\lambda_k)cc^T)x = \mu x$. On the other hand Theorem 2.2 guarantees the existence of an eigenvalue λ_k of (2.3) with this property. Hence, we have proved the following global existence and interlacing result.

THEOREM 2.6. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function of one sign such that* (2.5) *holds for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \ne \mu$. Then the nonlinear eigenvalue problem*

$$(A + \phi(\lambda)cc^T)x = \lambda x$$

has exactly n eigenvalues λ_k, $k = 1, \dots, n$.

The following interlacing properties are satisfied

$$\alpha_1 \le \lambda_1 \le \alpha_2 \le \lambda_2 \le \dots \le \alpha_n \le \lambda_n \quad \text{if } \phi(\lambda) \ge 0 \text{ for } \lambda \in \mathbb{R}, \tag{2.11}$$

and

$$\lambda_1 \le \alpha_1 \le \lambda_2 \le \alpha_2 \le \dots \le \lambda_n \le \alpha_n \quad \text{if } \phi(\lambda) \le 0 \text{ for } \lambda \in \mathbb{R}, \tag{2.12}$$

Waiving the sign condition for ϕ we obtain with the same techniques:

THEOREM 2.7. *For $\phi \in C[\alpha_{k-1}, \alpha_{k+1}]$ the nonlinear eigenvalue problem (2.3) has an eigenvalue $\hat\lambda \in [\alpha_{k-1}, \alpha_{k+1}]$ which is the kth smallest eigenvalue of $(A + \phi(\hat\lambda)cc^T)x = \mu x$. It has at most one eigenvalue $\hat\lambda \in (\alpha_{k-1}, \alpha_{k+1})$ with this property if ϕ satisfies condition (2.5) in $[\alpha_{k-1}, \alpha_{k+1}]$.*

If $\phi \in C(\mathbb{R})$ satisfies condition (2.5) in $\mathbb{R}$, then (2.3) has exactly n eigenvalues $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n$, and $\alpha_{k-1} \le \lambda_k \le \alpha_{k+1}$.

3. Small rank perturbations. We now consider a perturbation of a symmetric eigenvalue problem

$$(A + \tau H)x = \lambda x \tag{3.1}$$

where $A, H \in \mathbb{R}^{n\times n}$ are symmetric, H has small rank $r << n$, and $\tau \in \mathbb{R}$. Again we denote by $\alpha_1 \le \ldots \le \alpha_n$ the eigenvalues of A, and we set $\alpha_j = -\infty$ for $j < 1$ and $\alpha_j = \infty$ for $j > n$. The inertia of H is denoted by (π, ν, ζ).

The following generalization of Theorem 2.1 can be found in the book of Parlett [8], Corollary 10.3.1, and for positive semidefinite H in Gantmacher [2].

THEOREM 3.1. *Let $\beta_1 \le \ldots \le \beta_n$ denote the eigenvalues of $B := A + \tau H$. Then it holds that*

$$\alpha_{i-\nu} \le \beta_i \le \alpha_{i+\pi} \quad \text{for } \tau > 0,\ i = 1, \ldots, n \tag{3.2}$$

$$\alpha_{i-\pi} \le \beta_i \le \alpha_{i+\nu} \quad \text{for } \tau < 0,\ i = 1, \ldots, n. \tag{3.3}$$

Theorem 3.1 immediately yields the following existence result for the nonlinear modification of A.

THEOREM 3.2. *For $k \in \{1, \ldots, n\}$ let $\phi \in C[\alpha_{k-\nu}, \alpha_{k+\pi}]$ be nonnegative. Then the nonlinear eigenvalue problem*

$$(A + \phi(\lambda)H)x = \lambda x \tag{3.4}$$

has an eigenvalue $\hat\lambda \in [\alpha_{k-\nu}, \alpha_{k+\pi}]$. $\hat\lambda$ is the kth smallest eigenvalue of the linear eigenvalue problem

$$(A + \phi(\lambda)H)x = \mu x \tag{3.5}$$

with $\lambda = \hat\lambda$.

Proof. If $\alpha_{k-\nu} = \alpha_{k+\pi}$ then the corresponding invariant subspace of A contains a vector x such that $Hx = 0$. Obviously, x is an eigenvector of (3.4) corresponding to the eigenvalue α_k.

For $\alpha_{k-\nu} < \alpha_{k+\pi}$ the kth smallest eigenvalue $\mu_k(\lambda)$, of problem (3.5) is contained in $[\alpha_{k-\nu}, \alpha_{k+\pi}]$. Hence, the continuous mapping $\lambda \mapsto \mu_k(\lambda)$ has a fixed point $\hat\lambda \in [\alpha_{k-\nu}, \alpha_{k+\pi}]$ which is an eigenvalue of (3.4). ∎

The uniqueness result obtains the following form:

THEOREM 3.3. *Assume that the conditions of Theorem 3.2 hold and that for some $\delta > 0$ the condition*

$$\frac{\phi(\lambda) - \phi(\mu)}{\lambda - \mu}\|H\|_2 \le 1 - \delta \tag{3.6}$$

is satisfied in $I := [\alpha_{k-\nu}, \alpha_{k+\pi}]$, where $\|H\|_2$ is the spectral norm of H. Then problem (3.4) has at most one eigenvalue $\hat\lambda \in (\alpha_{k-\nu}, \alpha_{k+\pi})$ which is the kth smallest eigenvalue of (3.5) with $\lambda = \hat\lambda$.

Proof. The proof follows the same lines as the one of Theorem 2.3. Let $\hat\lambda = \mu_k(\hat\lambda) \in (\alpha_{k-\nu}, \alpha_{k+\pi})$ be an eigenvalue of (3.4), and let $x(\lambda)$ be an eigenvector of (3.5) corresponding to $\mu_k(\lambda)$ with $\|x(\lambda)\| = 1$.

We assume that $\lambda \neq \hat\lambda$. Multiplying (3.4) for $\hat\lambda$ from the left by $x(\lambda)^T$, multiplying (3.5) for λ from the left by $x(\hat\lambda)^T$, and subtracting yields

$$(\phi(\lambda) - \phi(\hat\lambda))x(\lambda)^T Hx(\hat\lambda) = (\mu_k(\lambda) - \hat\lambda))x(\lambda)^T x(\hat\lambda).$$

For λ close to $\hat\lambda$ we may assume that $x(\lambda)^T x(\hat\lambda) > 0$, and it follows

$$\frac{\mu_k(\lambda) - \hat\lambda}{\lambda - \hat\lambda} = \frac{\phi(\lambda) - \phi(\hat\lambda)}{\lambda - \hat\lambda}\frac{x(\lambda)^T Hx(\hat\lambda)^T}{x(\lambda)^T x(\hat\lambda)}. \tag{3.7}$$

For $(\phi(\lambda) - \phi(\hat\lambda))/(\lambda - \hat\lambda) > 0$ we further obtain from (3.6) and the continuous dependence of $x(\lambda)$ on λ

$$\begin{aligned}\frac{\mu_k(\lambda) - \hat\lambda}{\lambda - \hat\lambda} &\leq \frac{\phi(\lambda) - \phi(\hat\lambda)}{\lambda - \hat\lambda}\|H\|\frac{\|x(\lambda)\| \cdot \|x(\hat\lambda)\|}{x(\lambda)^T x(\hat\lambda)} \\ &\leq (1-\delta)\frac{\|x(\lambda)\| \cdot \|x(\hat\lambda)\|}{x(\lambda)^T x(\hat\lambda)} < 1\end{aligned}$$

if λ is sufficiently close to $\hat\lambda$. Hence, for $(\phi(\lambda) - \phi(\hat\lambda))/(\lambda - \phi(\hat\lambda)) > 0$ (2.6) holds and for $(\phi(\lambda) - \hat\lambda)/(\lambda - \hat\lambda) \leq 0$ these inequalities are trivial. ☐

The following theorem describes the global behavior of the spectrum of the nonlinear eigenproblem (3.1).

THEOREM 3.4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function of one sign such that condition* (3.6) *holds for all $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq \mu$. Then the nonlinear eigenvalue problem*

$$(A + \phi(\lambda)H)x = \lambda x$$

has exactly n eigenvalues λ_k, $k = 1, \dots, n$.

The interval $[\alpha_k, \alpha_{k+1}]$ contains at most $\nu + \pi + 1$ eigenvalues λ_j where for nonnegative ϕ it holds that $j \in \{k-\nu, k-\nu+1, \dots, k+\pi-1, k+\pi\}$, and for nonpositive ϕ $j \in \{k-\pi, k-\pi+1, \dots, k+\nu-1, k+\nu\}$.

4. Numerical methods. In this section we discuss methods for computing an eigenpair $(\hat\lambda, \hat x)$ such that $\hat\lambda$ is the kth smallest eigenvalue of $A + \phi(\hat\lambda)cc^T$ and $A + \phi(\hat\lambda)H$, respectively. Under the conditions of Theorems 2.6 and 3.4 this is the kth smallest eigenvalue of the nonlinear eigenvalue problem (1.1), in general for $\phi \geq 0$ this is an eigenvalue in $[\alpha_k, \alpha_{k+1}]$ and $[\alpha_{k-\nu}, \alpha_{k+\pi}]$, respectively.

4.1. Rank 1 modification. We first consider the rank one modification of A

$$(A + \phi(\lambda)cc^T)x = \lambda x \tag{4.1}$$

where $\phi(\lambda) \geq 0$ in $[\alpha_k, \alpha_{k+1}]$ and property (2.5) is satisfied. For $\phi(\lambda) \leq 0$ the methods are modified in an obvious way.

For this task Huang, Bai and Su [4] studied three methods for the case $\phi'(\lambda) \leq 0$: a safe guarded (linearly convergent) Picard iteration, a safeguarded (quadratically convergent) Rayleigh quotient iteration, and the successive linear approximation method which (under the additional condition $\phi''(\lambda) \geq 0$) is shown to be monotonically increasing and quadratically convergent. Safeguarding was based on the fact that $\lambda \mapsto \mu_k(\lambda)$ is montonically decreasing (which does not hold true under condition (2.5)) and on Sylvester's inertia theorem.

An iteration step of our first method is based on the solution of the linear eigenvalue problem

$$(A + \phi(\tilde{\lambda})cc^T)x = \mu x \tag{4.2}$$

where $\tilde{\lambda}$ is the current approximation to $\hat{\lambda}$.

To guarantee the convergence of our method we base a safeguarding on the fact, that for $\lambda \in (\alpha_k, \alpha_{k+1})$ it holds that (cf. proof of Theorem 2.3)

$$\lambda \leq \hat{\lambda} \quad \Leftrightarrow \quad \mu_k(\lambda) \geq \lambda. \tag{4.3}$$

Given an interval $[\lambda_\ell, \lambda_u]$ which contains $\hat{\lambda}$ and an approximation $\tilde{\lambda} \in (\lambda_\ell, \lambda_u)$ to $\hat{\lambda}$, we determine the kth smallest eigenvalue $\mu_k(\tilde{\lambda})$ of (4.2). Depending on the sign of $\tilde{\lambda} - \mu_k(\tilde{\lambda})$ the bracketing interval $[\lambda_\ell, \lambda_u]$ is reduced.

Moreover, the solution of (4.2) allows for an error bound which comes for free.

LEMMA 4.1. *Assume that ϕ is differentiable in $I := [\lambda_\ell, \lambda_u]$, $\hat{\lambda}, \tilde{\lambda} \in (\lambda_\ell, \lambda_u)$, and $x(\tilde{\lambda})$ is an eigenvector of* (4.2) *corresponding to the kth smallest eigenvalue $\mu_k(\tilde{\lambda})$.*

If $\phi'(\lambda) \leq 0$ for $\lambda \in I$ then it holds that

$$|\tilde{\lambda} - \hat{\lambda}| \leq |\mu_k(\tilde{\lambda}) - \tilde{\lambda}|, \tag{4.4}$$

and in the general case

$$|\tilde{\lambda} - \hat{\lambda}| \leq \frac{1}{1-\gamma}|\mu_k(\tilde{\lambda}) - \tilde{\lambda}| \tag{4.5}$$

where

$$\gamma := \max\{0, \max_{\lambda \in I} \phi'(\lambda)\|c\|^2\} < 1.$$

Proof. Differentiating the defining equation of $(\mu_k(\tilde{\lambda}), x(\tilde{\lambda}))$ and multiplying by $x(\tilde{\lambda})^T$ from the left yields

$$\mu_k'(\tilde{\lambda}) = \phi'(\tilde{\lambda})\frac{(c^T x(\tilde{\lambda}))^2}{\|x(\tilde{\lambda})\|^2}.$$

Hence,

$$\mu_k(\tilde{\lambda}) - \hat{\lambda} = \mu_k(\tilde{\lambda}) - \mu_k(\hat{\lambda}) = \mu_k'(\xi)(\tilde{\lambda} - \hat{\lambda})$$

for some $\xi = \tilde{\lambda} + \theta(\hat{\lambda} - \tilde{\lambda})$, $\theta \in (0, 1)$.

If $\phi'(\xi) \leq 0$, then $\mu_k'(\xi) \leq 0$ such that

$$\tilde{\lambda} \leq \hat{\lambda} \quad \Longrightarrow \quad \mu_k(\tilde{\lambda}) \geq \hat{\lambda} \quad \Longrightarrow \quad \mu_k(\tilde{\lambda}) - \tilde{\lambda} \geq \hat{\lambda} - \tilde{\lambda} \geq 0$$

and

$$\tilde{\lambda} \geq \hat{\lambda} \quad \Longrightarrow \quad \mu_k(\tilde{\lambda}) \leq \hat{\lambda} \quad \Longrightarrow \quad \tilde{\lambda} - \mu_k(\tilde{\lambda}) \geq \tilde{\lambda} - \hat{\lambda} \geq 0.$$

Hence,

$$|\tilde{\lambda} - \hat{\lambda}| \leq |\mu_k(\tilde{\lambda}) - \tilde{\lambda}|,$$

and in particular (4.4) is shown.

If $\phi'(\xi) \geq 0$, then $0 \leq \mu_k'(\xi) \leq \phi'(\xi)\|c\|^2 \leq \gamma$, from which we obtain in case $\tilde{\lambda} \leq \hat{\lambda}$

$$\hat{\lambda} - \mu_k(\tilde{\lambda}) \leq \gamma(\hat{\lambda} - \tilde{\lambda}) \quad \Longrightarrow \quad \hat{\lambda} \leq \frac{\mu_k(\tilde{\lambda}) - \gamma\tilde{\lambda}}{1-\gamma} \quad \Longrightarrow \quad \hat{\lambda} - \tilde{\lambda} \leq \frac{\mu_k(\tilde{\lambda}) - \tilde{\lambda}}{1-\gamma}$$

and for $\hat{\lambda} \leq \tilde{\lambda}$

$$\mu_k(\tilde{\lambda}) - \hat{\lambda} \leq \gamma(\tilde{\lambda} - \hat{\lambda}) \quad \Longrightarrow \quad \hat{\lambda} \geq \frac{\mu_k(\tilde{\lambda}) - \gamma\tilde{\lambda}}{1-\gamma} \quad \Longrightarrow \quad \tilde{\lambda} - \hat{\lambda} \leq \frac{\tilde{\lambda} - \mu_k(\tilde{\lambda})}{1-\gamma},$$

and therefore

$$|\lambda - \hat{\lambda}| \leq \frac{|\mu_k(\lambda) - \lambda|}{1-\gamma}.$$

□

To determine a new approximation to $\hat{\lambda}$ we take advantage of the approximation properties of the Rayleigh functional which is implicitly defined by the equation

$$f(\lambda, x) := x^T(A + \phi(\lambda)cc^T - \lambda I)x = 0, \; x \in \mathbb{R}^n \setminus \{0\}. \tag{4.6}$$

LEMMA 4.2. *Assume that condition* (2.5) *holds in an interval* $I \subset \mathbb{R}$. *If* $f(\tilde{\lambda}, x) = 0$ *for some* $\tilde{\lambda} \in I$ *and* $x \neq 0$, *then it holds that*

$$(\lambda - \tilde{\lambda})f(\lambda, x) < 0 \quad \textit{for every } \lambda \in I, \; \lambda \neq \tilde{\lambda}. \tag{4.7}$$

Proof. For $\lambda \neq \tilde{\lambda}$

$$\begin{aligned}(\lambda - \tilde{\lambda})f(\lambda, x) &= (\lambda - \tilde{\lambda})(f(\lambda, x) - f(\tilde{\lambda}, x)) \\ &= (\lambda - \tilde{\lambda})x^T((\phi(\lambda) - \phi(\tilde{\lambda}))cc^T - (\lambda - \tilde{\lambda})I)x \\ &= (\lambda - \tilde{\lambda})^2 \left(\frac{\phi(\lambda) - \phi(\tilde{\lambda})}{\lambda - \tilde{\lambda}}(c^T x)^2 - x^T x\right). \end{aligned} \tag{4.8}$$

For $(\phi(\lambda) - \phi(\tilde{\lambda}))/(\lambda - \tilde{\lambda}) \leq 0$ the statement is obvious. Otherwise we get from (2.5)

$$(\lambda - \tilde{\lambda})f(\lambda, x) \leq (\lambda - \tilde{\lambda})^2 \left(\frac{\phi(\lambda) - \phi(\tilde{\lambda})}{\lambda - \tilde{\lambda}}\|c\|^2\|x\|^2 - \|x\|^2\right) \leq -\gamma(\lambda - \tilde{\lambda})^2\|x\|^2 < 0.$$

□

If $x(\tilde{\lambda})$ is an eigenvector of (4.2) and $f(\lambda_\ell, x_k(\tilde{\lambda}))f(\lambda_u, x_k(\tilde{\lambda})) < 0$ then it follows from Lemma 4.2 that $f(\lambda, x_k(\tilde{\lambda})) = 0$ has exactly one root. If so we continue with this root as new approximation to $\hat{\lambda}$, otherwise we use a bisection step and choose $0.5(\alpha_\ell + \alpha_u)$. This results in Algorithm 1.

REMARK 4.3. *If* ϕ *is nonincreasing on* $I := [\alpha_k, \alpha_{k+1}]$ *then it follows from* (4.8) *and* $(\phi(\lambda) - \phi(\tilde{\lambda}))/(\lambda - \tilde{\lambda}) \leq 0$ *that*

$$(\lambda - \tilde{\lambda})f(\lambda, x) \leq -(\lambda - \tilde{\lambda})^2\|x\|^2 < 0 \quad \textit{for } \lambda \neq \tilde{\lambda}.$$

Algorithm 1 This algorithm computes an eigenpair $(\hat\lambda, \hat x)$ of a rank-1-modification $B(\alpha) := A + \phi(\lambda)cc^T$ of a symmetric eigenvalue problem by a quadratically convergent method. $\hat\lambda$ is the kth smallest eigenvalue of $B(\hat\lambda)$

Require: initial bounds $\lambda_\ell := \alpha_k$, $\lambda_u := \alpha_{k+1}$ and initial guess $\lambda \in [\lambda_\ell, \lambda_u]$ of $\hat\lambda$
$\gamma = \max(0, \max_{\lambda\in[\lambda_\ell,\lambda_u]} \phi'(\lambda)\|c\|^2)$

1: determine an eigenpair (μ, x) corresponding to the kth smallest eigenvalue of

$$(A + \phi(\lambda)cc^T)x = \mu x$$

2: **while** $|\lambda - \mu|/(1-\gamma) > \text{tol}$ **do**
3:   **if** $\mu > \lambda$ **then**
4:     $\lambda_\ell = \lambda$
5:   **else**
6:     $\lambda_u = \lambda$
7:   **end if**
8:   **if** $f(\lambda_\ell, x)f(\lambda_u, x) > 0$ **then**
9:     $\lambda = 0.5(\lambda_u + \lambda_\ell)$
10:   **else**
11:     solve $x^T(A + \phi(\lambda)cc^T - \lambda I)x = 0$ for λ
12:   **end if**
13:   determine an eigenpair (μ, x) corresponding to the kth smallest eigenvalue of

$$(A + \phi(\lambda)cc^T)x = \mu x$$

14: **end while**
15: $\hat\lambda = \lambda$, $\hat x := x$

Hence, $f(\lambda_\ell, x) > 0 > f(\lambda_u, x)$*, and equation* (4.6) *has a unique solution in* $(\lambda_\ell, \lambda_u)$ *such that the bisection step in line 9 of Algorithm 1 never occurs.*

Due to the approximation properties of the Rayleigh functional Algorithm 1 converges quadratically.

THEOREM 4.4. *Assume that condition* (2.5) *holds and that* ϕ *is continuously differentiable in a neigborhood of* $\hat\lambda$*. Let* $x_k(\lambda)$ *be an eigenvector of* (4.2)*.*

Then there exists a neigborhood U *of* $\hat\lambda$ *such that for every* $\lambda \in U$ *the equation* $f(\nu, x_k(\lambda)) = 0$ *has a unique solution* $\nu = \psi(\lambda)$*, and* $\psi'(\hat\lambda) = 0$*.*

Proof. From (2.5) it follows that $I - \phi'(\hat\lambda)cc^T$ is positive definite. Thus,

$$\frac{\partial}{\partial\lambda} f(\hat\lambda, \hat x) = \hat x^T(\phi'(\hat\lambda)cc^T - I)\hat x \neq 0$$

and it follows from the implicit function theorem that $f(\lambda, x) = 0$ has a unique solution $\lambda = \lambda(x)$ close to $\hat\lambda$ for every x in a neigborhood of $\hat x$. In particular by the continuity of $x_k(\lambda)$ it follows that there exists a neigborhood U of $\hat\lambda$ and a function $\psi : U \to \mathbb{R}$ such that $f(\psi(\lambda), x_k(\lambda)) = 0$, and differentiation yields

$$\begin{aligned} 0 &= \frac{d}{d\lambda} f(\psi(\lambda), x_k(\lambda)) \\ &= 2x_k(\lambda)^T(A + \phi(\psi(\lambda))cc^T - \psi(\lambda)I)\dot x_k(\lambda) + x_k(\lambda)^T(\phi'(\psi(\lambda))cc^T - I)x_k(\lambda)\psi'(\lambda), \end{aligned}$$

from which we obtain

$$0 = \frac{d}{d\lambda} f(\hat{\lambda}, \hat{x}) = \hat{x}^T(\phi'(\hat{\lambda})cc^T - I)\hat{x}\psi'(\hat{\lambda}),$$

i.e. $\psi'(\hat{\lambda}) = 0$ □

We can even get cubic convergence if we replace $x_k(\lambda)$ in Algorithm 1 with one step of the successive linear approximation method (cf. [9, 12]), i.e. with the eigenvector $x_k(\lambda)$ corresponding to kth smallest eigenvalue $\nu_k(\lambda)$ of the generalized eigenvalue problem

$$(A + (\phi(\lambda) - \lambda\phi'(\lambda))cc^T)x = \nu_k(\lambda)(I - \phi'(\lambda)cc^T)x. \tag{4.9}$$

Obviously, the only fixed point of $\lambda \mapsto \nu_k(\lambda)$ in (α_k, α_{k+1}) is $\hat{\lambda}$. Hence, (4.3) holds again and the safeguarding in Algorithm 1 applies also for this choice of $\nu_k(\lambda)$.

Lemma 4.5 contains an easily computable error bound generalizing Lemma 4.1.

LEMMA 4.5. *Assume that ϕ is differentiable in $I = [\lambda_\ell, \lambda_u]$, $\tilde{\lambda} \in (\lambda_\ell, \lambda_u)$, and $x(\tilde{\lambda})$ is an eigenvalue of (4.9) corresponding to the kth smallest eigenvalue $\nu_k(\tilde{\lambda})$.*

Let

$$\min\{\nu_k(\tilde{\lambda}) - \alpha_k, \alpha_{k+1} - \nu_k(\tilde{\lambda})\} > \Gamma|\tilde{\lambda} - \nu_k(\tilde{\lambda})|, \tag{4.10}$$

with $\Gamma := \|c\|^2 \max_{\lambda\in I} |\phi'(\lambda)|$. Then it holds that

$$|\tilde{\lambda} - \hat{\lambda}| \leq \frac{1+\Gamma}{1-\gamma}|\nu_k(\tilde{\lambda}) - \tilde{\lambda}|. \tag{4.11}$$

If $\phi'(\lambda) \leq 0$ for $\lambda \in I$ then

$$|\tilde{\lambda} - \hat{\lambda}| \leq (1+\Gamma)|\nu_k(\tilde{\lambda}) - \tilde{\lambda}|. \tag{4.12}$$

Proof. From (4.9) it follows that

$$(A + \phi(\tilde{\lambda})cc^T)x(\tilde{\lambda}) - \nu_k(\tilde{\lambda})x(\tilde{\lambda}) = (\tilde{\lambda} - \nu_k(\tilde{\lambda}))\phi'(\tilde{\lambda})cc^T x(\tilde{\lambda}). \tag{4.13}$$

Hence

$$\|(A + \phi(\tilde{\lambda})cc^T)x(\tilde{\lambda}) - \nu_k(\tilde{\lambda})x(\tilde{\lambda})\| \leq \Gamma|\tilde{\lambda} - \nu_k(\tilde{\lambda})| \cdot \|x(\tilde{\lambda})\|,$$

and $A + \phi(\tilde{\lambda})cc^T$ has an eigenvalue $\tilde{\mu}(\tilde{\lambda})$ such that (cf. [8], p. 73)

$$|\nu_k(\tilde{\lambda}) - \tilde{\mu}(\tilde{\lambda})| \leq \Gamma|\tilde{\lambda} - \nu_k(\tilde{\lambda})|.$$

(4.10) yields that $\tilde{\mu}(\tilde{\lambda}) \in (\alpha_k, \alpha_{k+1})$, and $\tilde{\mu}(\tilde{\lambda})$ is the kth smallest eigenvalue of $A + \phi(\tilde{\lambda})cc^T$. Therefore, Lemma 4.1 yields

$$|\tilde{\lambda} - \hat{\lambda}| \leq \frac{|\tilde{\mu}(\tilde{\lambda}) - \tilde{\lambda}|}{1-\gamma} \leq \frac{|\tilde{\mu}(\tilde{\lambda}) - \nu_k(\tilde{\lambda})| + |\nu_k(\tilde{\lambda}) - \tilde{\lambda}|}{1-\gamma} \leq \frac{1+\Gamma}{1-\gamma}|\nu_k(\tilde{\lambda}) - \tilde{\lambda}|.$$

For $\phi'(\lambda) \leq 0$ (4.12) follows from (4.4). □

The cubic convergence of Algorithm 2 follows Theorem 4.6:

THEOREM 4.6. *Assume that condition (2.5) holds and that ϕ is twice continuously differentiable in a neigborhood of $\hat{\lambda}$. Let $x_k(\lambda)$ be an eigenvector of (4.9).*

Algorithm 2 This algorithm computes an eigenpair $(\hat{\lambda}, \hat{x})$ of a rank-1-modification $B(\lambda) := A + \phi(\lambda)cc^T$ of a symmetric eigenvalue problem by a cubically convergent method. $\hat{\lambda}$ is the kth smallest eigenvalue of $B(\hat{\lambda})$

Require: initial bounds $\lambda_\ell := \alpha_k$, $\lambda_u := \alpha_{k+1}$ and initial guess $\lambda \in [\alpha_\ell, \alpha_u]$ of $\hat{\lambda}$
  $\gamma = \max(0, \max_{\lambda\in[\lambda_\ell,\lambda_u]} \phi'(\lambda)\|c\|^2)$, $\Gamma = \max_{\lambda\in[\lambda_\ell,\lambda_u]} |\phi'(\lambda)|\|c\|^2$.

determine an eigenpair (ν, x) corresponding to the kth smallest eigenvalue of

$$\left(A + (\phi(\lambda) - \lambda\phi'(\lambda))cc^T\right) x = \nu(I - \phi'(\lambda)cc^T)x$$

while $(1+\Gamma)|\lambda - \nu|/(1-\gamma) > \text{tol}$ **do**
  if $\nu > \lambda$ **then**
    $\lambda_\ell = \lambda$
  else
    $\lambda_u = \lambda$
  end if
  if $f(\lambda_\ell, x) f(\lambda_u, x) > 0$ **then**
    $\lambda = 0.5(\lambda_u + \lambda_\ell)$
  else
    solve $x^T(A + \phi(\lambda)cc^T - \lambda I)x = 0$ for λ
  end if
  determine an eigenpair (ν, x) corresponding to the kth smallest eigenvalue of

$$\left(A + (\phi(\lambda) - \lambda\phi'(\lambda))cc^T\right) x = \nu(I - \phi'(\lambda)cc^T)x$$

end while
$\hat{\lambda} = \lambda$, $\hat{x} := x$

Then there exists a neigborhood U of $\hat{\lambda}$ such that for every $\lambda \in U$ the equation $f(\nu, x_k(\lambda)) = 0$ has a unique solution $\nu = \psi(\lambda)$, and $\psi'(\hat{\lambda}) = \psi''(\hat{\lambda}) = 0$.

Proof. $\psi'(\hat{\lambda}) = 0$ is obtained in an analogous way to the proof of Theorem 4.4.

Differentiating

$$(A + (\phi(\lambda) - \lambda\phi'(\lambda))cc^T)x(\lambda) = \nu(\lambda)(I - \phi'(\lambda)cc^T)x(\lambda)$$

and multiplying with $x(\lambda)^T$ from the left one easily gets $\nu'(\hat{\lambda}) = 0$ and

$$(A + \phi(\hat{\lambda})cc^T)x'(\hat{\lambda}) = \hat{\lambda}x'(\hat{\lambda}).$$

With these two facts the second derivative of the defining equation

$$x(\lambda)^T(A + \phi(\psi(\lambda))cc^T - \psi(\lambda)I)x(\lambda) = 0$$

at $\hat{\lambda}$ reduces to

$$x(\hat{\lambda})^T(\phi'(\hat{\lambda})cc^T - I)x(\hat{\lambda})\psi''(\hat{\lambda}) = 0$$

from which we get $\psi''(\hat{\lambda}) = 0$ since $\phi'(\hat{\lambda})cc^T - I$ is negative definit. ☐

4.2. Small rank modification. We now consider numerical methods for computing a kth eigenvalue $\hat{\lambda}$ and corresponding eigenvector $\hat{x}$ of

$$(A + \phi(\lambda)H)x = \lambda x, \tag{4.14}$$

i.e. an eigenvalue $\hat\lambda$ which is the kth smallest eigenvalue of the linear problem

$$(A + \phi(\hat\lambda)H)x = \mu x. \tag{4.15}$$

If (π, ν, ζ) denotes the inertia of H and ϕ is continuous and nonnegative on $\overline{I_k} := [\alpha_{k-\nu}, \alpha_{k+\pi}]$ then by Theorem 3.1 there exists a kth eigenvalue $\hat\lambda$ of (4.14) in $\overline{I_k}$, and if neither $\alpha_{k-\pi}$ nor $\alpha_{k+\pi}$ is a kth eigenvalue and if property (3.6) holds, then there exists a unique $\hat\lambda \in I_k := (\alpha_{k-\nu}, \alpha_{k+\pi})$.

The proof of Theorem 3.3 demonstrates that for $\lambda \in I_k$ it holds that

$$\lambda \le \hat\lambda \quad \Leftrightarrow \quad \mu_k(\lambda) \ge \lambda. \tag{4.16}$$

such that the same safeguarding as in Algorithm 1 applies.

The error bound 4.1 receives the following form

LEMMA 4.7. *Assume that ϕ is differentiable in $I := [\lambda_\ell, \lambda_u]$, $\hat\lambda, \tilde\lambda \in (\lambda_\ell, \lambda_u)$, and $x(\tilde\lambda)$ is an eigenvector of* (4.15) *corresponding to the kth smallest eigenvalue $\mu_k(\tilde\lambda)$.*

If $\phi'(\lambda) \le 0$ for $\lambda \in I$ then it holds that

$$|\tilde\lambda - \hat\lambda| \le |\mu_k(\tilde\lambda) - \tilde\lambda|, \tag{4.17}$$

and in the general case

$$|\tilde\lambda - \hat\lambda| \le \frac{1}{1-\gamma}|\mu_k(\tilde\lambda) - \tilde\lambda| \tag{4.18}$$

where

$$\gamma := \max\{0, \max_{\lambda \in I} \phi'(\lambda)\|H\|\} < 1.$$

An analogous proof as the one of Lemma 4.2 yields

LEMMA 4.8. *Assume that condition* (3.6) *holds in an interval $I \subset \mathbb{R}$. Let*

$$f(\lambda, x) := x^T(A + \phi(\lambda)H - \lambda I)x. \tag{4.19}$$

If $f(\tilde\lambda, x) = 0$ for some $\tilde\lambda \in I$ and $x \ne 0$, then it holds that

$$(\lambda - \tilde\lambda)f(\lambda, x) < 0 \quad \text{for every } \lambda \in I,\ \lambda \ne \tilde\lambda. \tag{4.20}$$

Hence an approximation to $\hat\lambda$ can be updated by the solution of $f(\lambda, x) = 0$, and the following generalization of Algorithm 1 results.

If ϕ is continuously differentiable in a neigborhood of $\hat\lambda$ then the local convergence of Algorithm 3 is quadratic, and the following generalization of Algorithm 2 converges even cubically if ϕ is twice continuously differentiable.

The error bound in Lemma 4.5 can not be generalized to the case $\text{rank}(H) > 1$ since $\tilde\mu(\tilde\lambda)$ constructed analogously as in the proof of Lemma 4.5 can not be shown to be the kth eigenvalue of (4.15). Since it is very likely that $\tilde\mu(\tilde\lambda)$ is the kth eigenvalue of (4.15) (at least if $\tilde\lambda$ is close to $\hat\lambda$) we use the bound analogous to (4.11) as termination criterion in Algorithm 4. Upon completion of the iteration one can make sure by one solve of the eigenvalue problem (4.15) and Lemma 4.7 that the kth eigenvalue of (4.14) has been found.

Algorithm 3 This algorithm computes an eigenpair $(\hat\lambda, \hat x)$ of a small rank modification $B(\lambda) := A + \phi(\lambda)H$ of a symmetric eigenvalue problem by a quadratically convergent method. $\hat\lambda$ is the kth smallest eigenvalue of $B(\hat\lambda)$

Require: initial bounds $\lambda_\ell := \alpha_{k-\nu}$, $\lambda_u := \alpha_{k+\pi}$ and initial guess $\lambda \in [\lambda_\ell, \lambda_u]$ of $\hat\lambda$
  $\gamma = \max(0, \max_{\lambda\in[\lambda_\ell,\lambda_u]} \phi'(\lambda)\|H\|)$

determine an eigenpair (μ, x) corresponding to the kth smallest eigenvalue of

$$(A + \phi(\lambda)H)x = \mu x$$

while $|\lambda - \mu|/(1-\gamma) > \mathrm{tol}$ **do**
  if $\mu > \lambda$ **then**
    $\lambda_\ell = \lambda$
  else
    $\lambda_u = \lambda$
  end if
  if $f(\lambda_\ell, x) f(\lambda_u, x) > 0$ **then**
    $\lambda = 0.5(\lambda_u + \lambda_\ell)$
  else
    solve $x^T(A + \phi(\lambda)H - \lambda I)x = 0$ for λ
  end if
  determine an eigenpair (μ, x) corresponding to the kth smallest eigenvalue of

$$(A + \phi(\lambda)H)x = \mu x$$

end while
$\hat\lambda = \lambda$, $\hat x := x$

5. Numerical examples. To demonstrate the efficiency of the theory and methods presented in the previous sections we consider three examples, two rank one and one rank three modification of the symmetric eigenvalue problem

$$Ax = \lambda Bx \tag{5.1}$$

with

$$A = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}, \quad B = \frac{h}{6}\begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}. \tag{5.2}$$

This is a finite element discretization of the eigenvalue problem

$$-u''(x) = \lambda u(x),\ x \in (0,1),\ u(0) = 0,\ u'(1) = 0 \tag{5.3}$$

with linear elements on a uniform grid.

In all three cases we considered $h = 0.01$ and a problem of dimension 100, and we terminated the iteration if the error of the wanted eigenvalue was less than $\mathrm{tol} = 10^{-8}$. The initial approximation of an eigenvalue in $[\alpha_{k-\nu}, \alpha_{k+\pi}]$ ($\nu = 0$ and $\pi = 1$ for rank one modifications) we always chose $\lambda = \alpha_{k-\nu}$. We never observed that a bisection

Algorithm 4 This algorithm computes an eigenpair $(\hat{\lambda}, \hat{x})$ of a small rank modification $B(\lambda) := A + \phi(\lambda)H$ of a symmetric eigenvalue problem by a cubically convergent method. $\hat{\lambda}$ is the kth smallest eigenvalue of $B(\hat{\lambda})$

Require: initial bounds $\lambda_\ell := \alpha_{k-\nu}$, $\lambda_u := \alpha_{k+\pi}$ and initial guess $\lambda \in [\alpha_\ell, \alpha_u]$ of $\hat{\lambda}$
 $\gamma = \max(0, \max_{\lambda\in[\lambda_\ell,\lambda_u]} \phi'(\lambda)\|H\|)$, $\Gamma = \max_{\lambda\in[\lambda_\ell,\lambda_u]} |\phi'(\lambda)|\|H\|$.

determine an eigenpair (ν, x) corresponding to the kth smallest eigenvalue of

$$(A + (\phi(\lambda) - \lambda\phi'(\lambda))H)\, x = \nu(I - \phi'(\lambda)H)x$$

while $(1 + \Gamma)|\lambda - \nu|/(1 - \gamma) > \text{tol}$ **do**
  if $\nu > \lambda$ **then**
    $\lambda_\ell = \lambda$
  else
    $\lambda_u = \lambda$
  end if
  if $f(\lambda_\ell, x)f(\lambda_u, x) > 0$ **then**
    $\lambda = 0.5(\lambda_u + \lambda_\ell)$
  else
    solve $x^T(A + \phi(\lambda)H - \lambda I)x = 0$ for λ
  end if
  determine an eigenpair (ν, x) corresponding to the kth smallest eigenvalue of

$$(A + (\phi(\lambda) - \lambda\phi'(\lambda))H)\, x = \nu(I - \phi'(\lambda)H)x$$

end while
$\hat{\lambda} = \lambda$, $\hat{x} := x$

step was necessary. All computations were done on a Pentium P Extreme Edition 955 with 3.4 GHz and 8 GB RAM under MATLAB 2008 b.

EXAMPLE 5.1. *Example 5.1 is identical to Example 2 in the paper of Huang, Bai and Su [4]. It models a string with a load attached to its end by an elastic spring. Here the right boundary condition of* (5.3) *has to be replaced by* $-u'(1) = \phi(\lambda)u(1)$ *resulting in the rank one modification*

$$\left(A + \phi(\lambda)cc^T\right) x = \lambda Bx \tag{5.4}$$

of (5.1)*, where* $c = (0, \dots, 0, 1)^T$ *and* $\phi(\lambda) = \frac{\lambda}{\lambda-1}$.

Tables 5.1 and 5.2 contain approximations to the five smallest eigenvalues of problem (5.4)*, the interval* $[\alpha_k, \alpha_{k+1}]$ *which contains the kth eigenvalue, and the residual norm* $\|Ax+\phi(\lambda)cc^Tx-\lambda Bx\|/\|x\|$*, as well as the number of iterations for the quadratically and cubically convergent method in Table 5.1 and 5.2, respectively.*

k	λ	iter	(α_k, α_{k+1})	res
1	4.48217654588734	4	(2.46745, 22.2107)	8.257067e-11
2	24.2235731125729	3	(22.2107, 61.7167)	6.432557e-11
3	63.7238211419571	3	(61.7167, 121.024)	8.938836e-11
4	123.031221068060	2	(121.024, 200.192)	4.437259e-10
5	202.200899143597	2	(200.192, 299.299)	9.148373e-11

TABLE 5.1
Rank one modification with algorithm 1

k	λ	iter	(α_k, α_{k+1})	res
1	4.48217654593128	3	(2.46745, 22.2107)	1.168638e-10
2	24.2235731125745	3	(22.2107, 61.7167)	6.272389e-11
3	63.7238211419634	2	(61.7167, 121.024)	5.747992e-11
4	123.031221067615	2	(121.024, 200.192)	7.687618e-11
5	202.200899143567	2	(200.192, 299.299)	7.567725e-11

TABLE 5.2
Rank one modification with algorithm 2

EXAMPLE 5.2. *We consider the rank one modification*

$$\left(A + \phi(\lambda)cc^T\right)x = \lambda Bx \tag{5.5}$$

of (5.1) *where c is a random vector and $\phi(\lambda) = 0.9(1 - \sin(\lambda))$. Here ϕ does not satisfy $\phi'(\lambda) \leq 0$ but only condition* (2.5).

Tables 5.3 and 5.4 are organized in the same way as in Example 5.1.

k	λ	iter	(α_k, α_{k+1})	res
1	3.60865562452412	4	(2.46745, 22.2107)	7.027947e-11
2	22.3266119553748	3	(22.2107, 61.7167)	9.194079e-11
3	61.7815514029768	3	(61.7167, 121.024)	1.018320e-10
4	121.024553173989	2	(121.024, 200.192)	1.245541e-10
5	200.212089249125	3	(200.192, 299.299)	1.113477e-10

TABLE 5.3
Rank one modification with algorithm 1

k	λ	iter	(α_k, α_{k+1})	res
1	3.60865562451013	3	(2.46745, 22.2107)	8.367172e-11
2	22.3266119553773	3	(22.2107, 61.7167)	7.457378e-11
3	61.7815514029831	3	(61.7167, 121.024)	6.692810e-11
4	121.024553173988	2	(121.024, 200.192)	6.982292e-11
5	200.212089249125	2	(200.192, 299.299)	7.957128e-11

TABLE 5.4
Rank one modification with algorithm 2

EXAMPLE 5.3. *Consider the nonlinear modification*

$$(A + \phi(\lambda)H)\, x = \lambda Bx \tag{5.6}$$

of (5.1) *where H is a symmetric rank three matrix with one negative and two positive eigenvalues. ϕ is again $\phi(\lambda) = 0.9(1 - \sin(\lambda))$ and $\max \phi'(\lambda)\|H\| = 0.996$ such that Theorem 3.2 applies. Although the initial value $\lambda = \alpha_{k-\nu}$ is quite far away from the eigenvalue under consideration the method has no problem to converge and no bisection steps are needed.*

k	λ	iter	$(\alpha_{k-1}, \alpha_{k+2})$	res
1	3.33307908052349	3	(-Inf, 61.7167)	6.602111e-11
2	23.6619469468822	3	(2.46745, 121.024)	8.431452e-11
3	61.5752941605781	3	(22.2107, 200.192)	1.112121e-10
4	121.024515451116	2	(61.7167, 299.299)	8.898374e-11
5	200.192841286577	2	(121.024, 418.441)	8.610105e-11

TABLE 5.5
Rank three modification with algorithm 3

k	λ	iter	$(\alpha_{k-1}, \alpha_{k+2})$	res
1	3.33307908050168	3	(-Inf, 61.7167)	8.159673e-11
2	23.66194694688183	3	(2.46745, 121.024)	6.468610e-11
3	61.57529416011987	2	(22.2107, 200.192)	4.530087e-10
4	121.02451545111705	2	(61.7167, 299.299)	7.704706e-11
5	200.19284128658603	2	(121.024, 418.441)	6.600825e-11

TABLE 5.6
Rank three modification with algorithm 4

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