AN UNSYMMETRIC EIGENPROBLEM GOVERNING VIBRATIONS OF A PLATE WITH ATTACHED LOADS

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Abstract
This paper presents a structure preserving iterative projection method for the unsymmetric eigenvalue problem governing free vibrations of a plate with attached masses. A numerical example demonstrates the efficiency of the approach.

Keywords: unsymmetric eigenvalue problem, rational eigenproblem, fluid–solid interaction, variational characterisation of eigenvalues, iterative projection method, nonlinear Arnoldi method, Jacobi–Davidson method

1 Introduction

Free vibrations of a plate with elastically attached loads are governed by the unsymmetric eigenvalue problem

$$Lu(x) = \lambda \rho d u(x) + \lambda \sum_{j=1}^{\ell} m_j c_j \delta_{x-x_j} \quad , \quad x \in \Omega \tag{1}$$

$$u(x) = \frac{\partial}{\partial n} u(x) = 0 \quad , \quad x \in \partial\Omega \tag{2}$$

$$-\lambda m_j c_j + k_j (c_j - u(x_j)) = 0 \quad , \quad j = 1, \dots, \ell. \tag{3}$$

Here $\Omega \subset \mathbb{R}^2$ is a domain occupied by the plate, L is the plate operator. ρ is the mass per volume density, and d the thickness of the plate. For $j = 1, \dots, \ell$ at $x_j \in \Omega$ a load m_j is joined elastically to the plate with stiffness coefficient k_j, and c_j denotes the displacement of the mass m_j.

Although the eigenvalue problem (1)—(3) is unsymmetric we may expect real eigenvalues since free vibrations of a system are modelled, and indeed the realness of the spectrum of (1)—(3) can be proved in different ways.

Eliminating c_j one obtains a rational eigenvalue problem

$$Lu(x) = \lambda\rho d u(x) + \sum_{j=1}^{\ell} \frac{\lambda\sigma_j}{\sigma_j - \lambda} m_j \delta_{x-x_j} u \quad , \quad x \in \Omega \tag{4}$$

$$u(x) = \frac{\partial}{\partial n} u(x) = 0 \quad , \quad x \in \partial\Omega \tag{5}$$

where $\sigma_j = k_j/m_j$. Embedding problem (4), (5) into a family of linear eigenvalue problems it was shown in [1, 2] that (1)—(3) has a countable set of real and positive eigenvalues.

A much easier way to deduce the realness of the spectrum of (1)—(3) is the following. Let λ be an eigenvalue with corresponding eigenfunction (u, c). Multiplying (1) by $\overline{u}$ and integrating over Ω, and multiplying (3) by $-\lambda c^H D_m D_k^{-1}$ with $D_k = \text{diag}\{k_j\}$ and $D_m = \text{diag}\{m_j\}$, and summing up one gets the quadratic equation

$$-\lambda^2 c^H D_m D_k^{-1} D_m c - \lambda(c^H D_k^{-1} D_m D_k^{-1} c - \int_\Omega d\rho |u|^2 \, dx) + \int_\Omega \overline{u} Lu \, dx = 0. \tag{6}$$

From $-c^H D_m D_k^{-1} D_m c < 0 < \int_\Omega \overline{u} Lu \, dx$ it follows that (6) has two real equations of opposite signs, and assuming that $\lambda < 0$ one gets a contradiction from (1) and (3).

Discretising (4) by finite elements one obtains the unsymmetric matrix eigenvalue problem

$$\begin{pmatrix} D_k & -D_k C \\ O & K_p \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \lambda \begin{pmatrix} D_m & O \\ C^T D_m & M_p \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix}, \tag{7}$$

where $K_p \in \mathbb{R}^{n\times n}$ and $M_p \in \mathbb{R}^{n\times n}$ are the stiffness and mass matrix of the plate, respectively, and $C \in \mathbb{R}^{n\times \ell}$ represents the coupling of the loads to the plate, i.e. the jth column is a unit vector with a one in the component corresponding to x_j. Similar to (1)—(3) it can be shown that all eigenvalues of (7) are real and positive.

Solving problem (7) by an iterative projection method like shift-and-invert Arnoldi or a rational Krylov method the special structure that guarantees the realness of the spectrum is destroyed and even non-real eigenvalue approximations may appear. In this paper we suggest a structure preserving projection approach based on the nonlinear Arnoldi method. Taking advantage of a minmax characterization of its eigenvalues we can compute the eigenvalues one after the other in a safe way. Comparing our method to the shift and invert Arnoldi method and the nonlinear Arnoldi method for the discretised version of the rational eigenvalue problem (4), (5) considered in [3] demonstrates its efficiency.

2 Variational characterization of eigenvalues

Multiplying the first equation of (7) by $D_m D_k^{-1}$ the eigenproblem receives the more convenient form

$$K\begin{pmatrix} y \\ u \end{pmatrix} := \begin{pmatrix} D_m & -D_m C \\ O & K_p \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix} = \lambda \begin{pmatrix} D_m D_k^{-1} D_m & O \\ C^T D_m & M_p \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix} =: \lambda M \begin{pmatrix} y \\ u \end{pmatrix}. \tag{8}$$

Unsymmetric eigenvalue problems of this form also describe free vibrations of fluid solid structures [4, 5].

We studied problem (8) in [6]. Taking advantage of the fact that $(y^T, u^T)^T$ is a right eigenvector of (8) corresponding to the eigenvalue λ if and only if $(\lambda y^T, u^T)^T$ is a left eigenvector we defined a Rayleigh functional

$$p(y,u) := \begin{cases} q(y,u) + \sqrt{q(y,u)^2 + \frac{u^T K_p u}{y^T D_m D_k^{-1} D_m y}} & \text{if} \quad y \neq 0 \\ \frac{u^T K_p u}{u^T M_p u} & \text{if} \quad y = 0 \end{cases} \tag{9}$$

with

$$q(z,u) := \frac{y^T D_m y - u^T M_p u - 2y^T D_m C u}{2y^T D_m D_k^{-1} D_m y}, \tag{10}$$

and we proved that right eigenvectors of (8) are stationary points of p, and that the following minmax characterization holds:

THEOREM 1 Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n+\ell}$ be the eigenvalues of problem (8), and let $(y_1, u_1), (y_2, u_2), \ldots$ corresponding right eigenvectors. Then it holds that

$$\lambda_k = \min_{\dim S_k = k} \max_{0 \neq (y,u) \in S_k} p(y,u). \tag{11}$$

This characterization can be obtained in a completely different way. Multiplying the first block equation of (8) by λ we obtain the quadratic eigenvalue problem

$$Q(\lambda)\begin{pmatrix} z \\ u \end{pmatrix} := \begin{pmatrix} \lambda^2 D_m D_k^{-1} D_m - \lambda D_m & \lambda D_m C \\ \lambda C^T D_m & \lambda M_p - K_p \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{12}$$

Obviously, $Q(\lambda)$ is symmetric, and the congruence transformation with

$$T := \begin{pmatrix} I & (\lambda\Delta - D_m)^{-1} D_m C \\ O & I \end{pmatrix}, \quad \Delta := D_m D_k^{-1} D_m$$

yields

$$T^T Q(\lambda) T = \begin{pmatrix} \lambda^2 \Delta - \lambda D_m & O \\ O & \lambda M_p - K_p - \lambda C^T D_m (\lambda\Delta - D_m)^{-1} D_m C \end{pmatrix}, \tag{13}$$

from which we immediately see that $Q(\lambda)$ is negative definite for sufficiently small positive λ, and it is positive definite for sufficiently large positive λ.

Hence, for every $(y, u) \neq (0, 0)$ the quadratic equation

$$f(\lambda; y, u) := (y^T, u^T)Q(\lambda)\begin{pmatrix} y \\ u \end{pmatrix} = 0$$

has exactly one positive solution, and it is easily seen that this is exactly the value $p(y, u)$ of the Rayleigh functional p given in (9), (10). Since

$$\frac{\partial}{\partial\lambda} f(\lambda; y, u)\Big|_{\lambda=p(y,u)} > 0,$$

$Q(\lambda)$is overdamped with respect to the interval $J := (0, \infty)$ with Rayleigh functional p [7, 8]. Hence, it has exactly $n + \ell$ eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n+\ell}$ which satisfy the minmax characterization (11).

3 Structure preserving iterative projection method

Since an essential part of the eigenvalue problem (8) is a discretization of a partial differential operator the eigenvalue problem (8) will be large and sparse, and hence a reasonable solution approach is an iterative projection method. Here the eigenproblem is projected to a subspace of small dimension which yields approximate eigenpairs. If an error tolerance is not met then the search space is expanded in an iterative way with the aim that some of the eigenvalues of the reduced matrix become good approximations of some of the wanted eigenvalues of the given large pencil.

If $V := \begin{pmatrix} V_m \\ V_p \end{pmatrix} \in \mathbb{R}^{\ell+n\times k}$ is a basis of an ansatz space the projection of problem (8) reads

$$\begin{aligned}
&(V_m^T D_m V_m - V_m^T D_m C V_p + V_p^T K_p V_p)z \\
&\quad = \quad \lambda(V_m^T D_m D_k^{-1} D_m V_m + V_p^T C^T D_m V_m + V_p^T M_p V_p)z.
\end{aligned}$$

Hence, the structure of problem (8) gets lost, and it is not certain that eigenvalues of the projected problem stay real. This suggests to use a structure preserving projection method, i.e. to project the mass part and the plate part of the problem individually to search spaces. Then the eigenvalues of the projected problem will be real, and from the minmax characterization we obtain that the eigenvalues of the projected problem yield upper bounds of the eigenvalues of the original problem (8):

THEOREM 2

Assume that $V = \begin{pmatrix} V_m & 0 \\ 0 & V_p \end{pmatrix} \in \mathbb{R}^{\ell+n\times k}$ has rank k. Let

$$K_V \;:=\; V^T \begin{pmatrix} D_m & -D_m C \\ O & K_p \end{pmatrix} V = \begin{pmatrix} V_m^T D_m V_m & -V_m^T D_m C V_p \\ 0 & V_p^T K_p V_p \end{pmatrix}, \tag{14}$$

$$M_V \;:=\; V^T \begin{pmatrix} D_m D_k^{-1} D_m & O \\ C^T D_m & M_p \end{pmatrix} V = \begin{pmatrix} V_m^T D_m D_k^{-1} D_m V_m & 0 \\ V_p^T C^T D_m V_m & V_p^T M_p V_p \end{pmatrix}, \tag{15}$$

and let $\tilde\lambda_1 \le \tilde\lambda_2 \dots \le \tilde\lambda_k$ be the eigenvalues of the projected eigenvalue problem

$$K_V z = \tilde\lambda M_V z. \tag{16}$$

Then it holds that

$$\lambda_j \le \tilde\lambda_j, \quad j = 1, 2, \dots, k. \tag{17}$$

PROOF

The projected problem (16) preserves the structure of (8), and therefore its eigenvalues satisfy a minmax characterization with respect to the Rayleigh functional $\tilde p$ which is defined conformable to (9), (10).

Let

$$z = \begin{pmatrix} y \\ u \end{pmatrix} \quad \text{and} \quad x := Vz = \begin{pmatrix} V_m y \\ V_p u \end{pmatrix}.$$

Then it is obvious that $\tilde p(z) = p(x)$. Hence, for $j = 1, \dots, k$ it holds that

$$\begin{aligned} \lambda_j &= \min_{\dim W = j, W \subset \mathbb{R}^{n+\ell}} \max_{x \in W, x \ne 0} p(x) \le \min_{\dim Z = j, Z \subset \mathbb{R}^k} \max_{x \in VZ, x \ne 0} p(x) \\ &= \min_{\dim Z = j, Z \subset \mathbb{R}^k} \max_{z \in Z, z \ne 0} p(Vz) = \min_{\dim Z = j, Z \subset \mathbb{R}^k} \max_{z \in Z, z \ne 0} \tilde p(z) = \tilde\lambda_j. \qquad \square \end{aligned}$$

To approximate all eigenvalues which do not exceed a given limit we determine them one after the other beginning with the smallest one. Assume that approximations to $\lambda_1 \le \dots \lambda_{m-1}$ and the corresponding eigenvectors have been obtained from the projection of (8) to $V := \operatorname{diag}\{V_m, V_p\}$ which satisfy a specified error tolerance, but the mth eigenpair does not. To improve the approximation properties of the projection method with respect to λ_m, it is reasonable to expand the ansatz space V by the direction of inverse iteration aiming at the eigenvector corresponding to λ_m, i.e.

$$t_{ii} = (K - \sigma M)^{-1} M x \tag{18}$$

where (σ, x) is an approximation to the mth eigenpair.

In the course of the algorithm the approximation σ to λ_m changes in every step, and therefore linear systems with varying system matrices have to be solved in consecutive iteration steps which is much to costly for truly large problems. We therefore keep σ fixed as long as possible, and to increase the robustness of the method (cf. [9, 10]) we replace t_{ii} by the Cayley transformation

$$t_{Ct} = (K - \sigma M)^{-1}(K - \theta M)x \tag{19}$$

where θ is the current approximation of the desired eigenvalue. Notice that it holds that

$$(\theta - \sigma)t_{ii} = (\theta - \sigma)(K - \sigma M)^{-1} M x = x - (K - \sigma M)^{-1}(K - \theta M)x,$$

and since the current Ritz vector x is already contained in V the expansions by t_{ii} and t_{Ct} are equivalent.

We arrive at the iterative projection method of Arnoldi type in Algorithm 1:

Algorithm 1 Arnoldi type iterative projection method

Require: Initial basis $V = \begin{pmatrix} V_m & O \\ O & V_p \end{pmatrix}$, $V_m^T V_m = I$, $V_p^T V_p = I$; $m = 1$; θ_m=0;

1: determine preconditioner $L \approx (K - \sigma M)^{-1}$, σ close to first wanted eigenvalue
2: **while** $\theta_m \leq$ maxeig **do**
3: solve the projected problem

$$\begin{pmatrix} V_m^T D_m V_m & -V_m^T D_m C V_p \\ 0 & V_p^T K_p V_p \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} = \theta \begin{pmatrix} V_m^T D_m D_k^{-1} D_m V_m & 0 \\ V_p^T C^T D_m V_m & V_p^T M_p V_p \end{pmatrix} \begin{pmatrix} y \\ u \end{pmatrix} \tag{20}$$

4: choose the m smallest eigenvalue θ_m and corresp. eigenvector $(y_m^T, u_m^T)^T$
5: determine Ritz vector $x = \begin{pmatrix} V_m y_m \\ V_p u_m \end{pmatrix}$ and residual $r = (K - \theta_m M)x$
6: **if** $\|r\|/\|x\| < \epsilon$ **then**
7: **while** $\|r\|/\|x\| < \epsilon$ **do**
8: accept approximate mth eigenpair (θ_m, x); increase $m \leftarrow m + 1$;
9: choose m smallest eigenvalue θ_m and corresp. eigenvector $(y_m^T, u_m^T)^T$
10: determine Ritz vector $x = \begin{pmatrix} V_m y_m \\ V_p u_m \end{pmatrix}$ and residual $r = (K - \theta_m M)x$
11: **end while**
12: reduce search space V if indicated
13: determine new preconditioner $L \approx (K - \theta M)^{-1}$ if necessary
14: **end if**
15: solve $Lt = r$ for $t = (t_m^T, t_p^T)^T$
16: orthogonalise $v_m = t_m - V_m V_m^T t_m$, $v_p = t_p - V_p V_p^T t_p$
17: if $\|v_m\| >$ tol expand $V_m \leftarrow [V_m, v_m/\|v_m\|]$
18: if $\|v_p\| >$ tol expand $V_p \leftarrow [V_p, v_p/\|v_p\|]$
19: update projected problem (20)
20: **end while**

Some comments are in order:

(i) Since the dimension of the projected eigenproblem is quite small it is solved by a dense solver and therefore approximations to further eigenpairs are at hand without additional cost.

(ii) In steps 7 to 11 we check whether approximations to further eigenpairs already satisfy the specified error tolerance. Moreover, at the end of the while-loop an approximation to the next eigenpair to compute and the residual is provided.

(iii) If the dimension of the search space has become too large we reduce the matrices V_m and V_p in step 12 such that the columns of V_m and V_p form an orthogonal basis of the space spanned by the mass and the plate part of the eigenvectors found so far. Notice, that the search space is reduced only after an eigenpair has converged because the reduction spoils too much information and the convergence can be retarded.

(iv) The preconditioner L is updated in step 13 if the convergence measured by the reduction of the residual norm has become too slow.

(v) For the unsymmetric eigenvalue problems governing free vibrations of fluid–solid structures a different expansion t of the search space was considered in [6], which is based on Jacobi–Davidson type arguments.

4 Numerical example

Consider the clamped plate occupying the domain $\Omega = (0,4) \times (0,3)$ with constant density $\rho = 1$ and constant thickness $d = 1$. We assume that 6 masses are attached to the plate at $x_1 = (1,1)$, $x_2 = (2,1)$, $x_3 = (3,1)$, $x_4 = (1,2)$, $x_5 = (2,2)$ and $x_6 = (3,2)$, where $\sigma_1 = \sigma_2 = \sigma_3 = 1000$, $\sigma_4 = \sigma_6 = 2000$, and $\sigma_5 = 3000$, and $m_1 = 1$, $m_2 = 1/2$ and $m_3 = 1/3$.

We discretised the eigenproblem by Bogner-Fox-Schmit elements on a quadratic mesh with step size $h = 0.05$ which yielded a matrix eigenvalue problem (7) of dimension 18650. The problem has 56 eigenvalues not exceeding 5000.

We determined these eigenvalues on an Intel Pentium D CPU at 3.2 GHZ with 3.5 GB RAM under MATLAB 7.7.0 (R2008b) with the structure preserving Arnoldi type iterative projection method of Section 3.

Initializing with a random vector the method required 19.8 seconds to determine all eigenvalues less than 5. An eigenpair was accepted if the relativ residual norm was less than 10^{-7}. We did not reduce the size of the search space, and we did not recycle the preconditioner, but we implemented the solution of $Lt = r$ in step 15 with the LU factorisation of $K - 1.5M$.

The method behaved as follows. Due to the bad accuracy of the initial vector the method constructed a search space of dimension 77 until it found the smallest eigenvalue. This space contained already so much information about the subsequent eigenvectors that for each of the 2nd to 6th eigenpair only expansions by three vectors were necessary. Expanding V by two more vectors we obtained the 7th, 8th, and 9th eigenpair with the desired accuracy from the same search space of dimension 94.

Only one or even no additional vector was necessary for the following eigenpairs. For example, we derived the 15th to 20th eigenvalue from a 99 dimensional space, and after expanding it by one vector we obtained the 21th to 28th eigenpair. The maximum dimension of the search space was 126.

In 25 cases we received eigenpair approximations with sufficient accuracy in steps 7 to 11 without expanding the search space V.

Figure 1 contains the convergence history of the method showing the relative residual norms $\|r\|/\|x\|$ for all residuals appearing in lines 5 and 10. The red line indicates the desired accuracy.

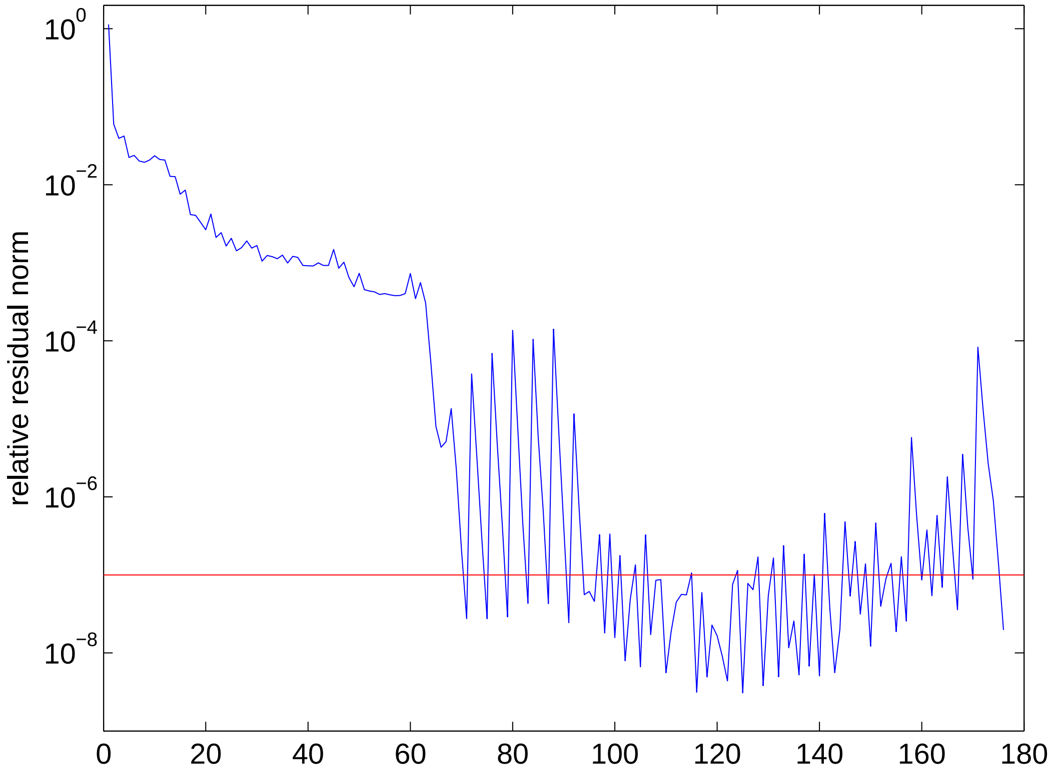

Figure 1: Convergence history of the iterative projection method of Arnoldi type for the plate problem in Section 4 with random initial vector

Choosing an initial vector of much higher accuracy does not change the situation very much. Figure 2 shows the convergence behaviour of the method for an initial vector which approximates the eigenvector corresponding to the smallest eigenvalue up to a relative error $3.0e-5$.

The method requires 20.4 seconds. The smallest eigenpair is obtained after 3 steps, but then a search space of dimension 77 is needed to find the second eigenpair with required accuracy. For the following eigenpairs the method behaves as for the random initial vector.

Replacing the expansion of the search space in step 15 of Algorithm 1 in a Jacobi–Davidson manner, i.e. by a coarse approximation to the solution of the correction equation

$$\left(1-\frac{xx^T}{x^Tx}\right)(K-\theta M)\left(1-\frac{xx^T}{x^Tx}\right)t=-(K-\theta M)x \tag{21}$$

where (θ,x) is the current approximation to an eigenpair the iterative projection method requires 32.4 seconds.

We compared our method with two existing approaches. We solved the corresponding discretisation of the rational eigenvalue problem (4), (5) with the nonlinear Arnoldi method discussed in [3] which needed 79.0 seconds, and with the shift-and-invert Arnoldi method implemented in the MATLAB function *sptarn* which required

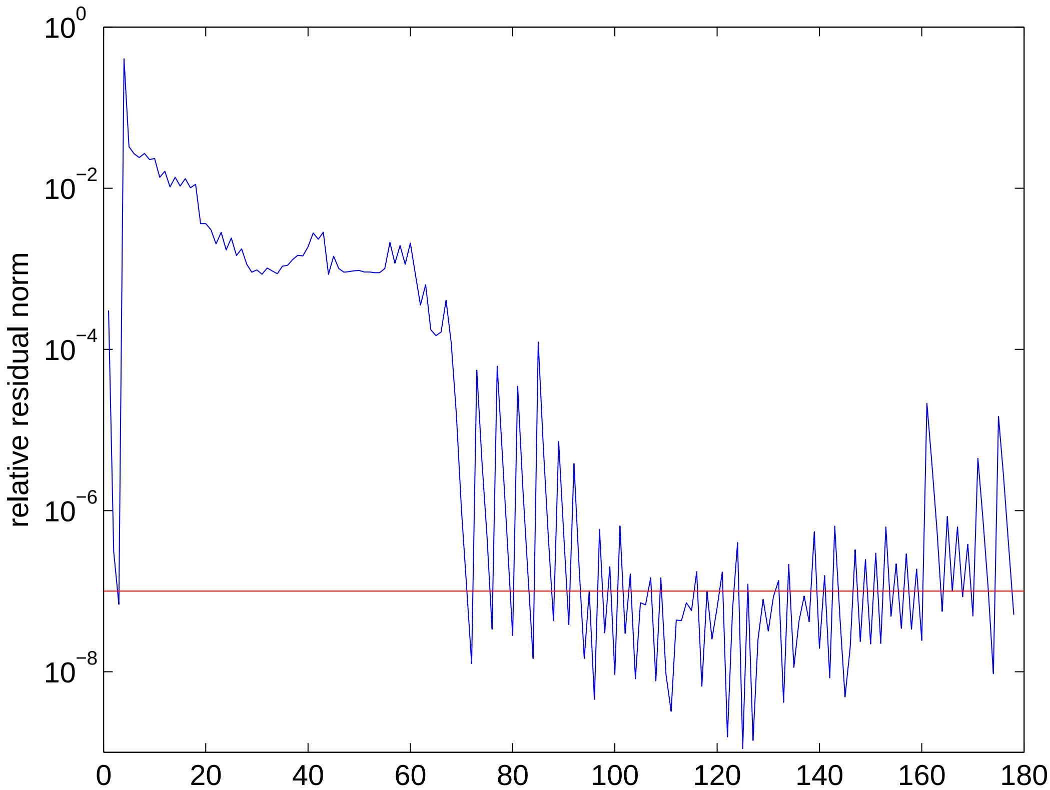

Figure 2: Convergence history of the iterative projection method of Arnoldi type for the plate problem in Section 4 with accurate initial vector

106.1 seconds.

5 Conclusions

For the unsymmetric linear eigenvalue problem governing free vibrations of a plate with attached loads we presented an Arnoldi type iterative projection method which is based on a minmax characterization of its eigenvalues. A numerical example demonstrates that the method is much faster than the shift-and-invert Arnoldi method for general unsymmetric problems and the nonlinear Arnoldi method for an equivalent rational eigenvalue problem.

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