On the existence and uniqueness of non-Newtonian shear-dependent flow in compliant vessels

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Abstract

We generalize the existence and uniqueness results obtained in [9, Filo, Zaušková] for the Navier-Stokes equation in a time dependent domain for the shear-dependent non-Newtonian fluids. The fluid equations are coupled with the generalized string equation describing domain deformation. Having now the non-linear viscous terms new techniques have to be used to obtain existence and uniqueness of the weak solution for non-Newtonian fluids.

keywords: non-Newtonian fluids, fluid-structure interaction, shear thinning fluids, shear-thickening fluids, hemodynamics, existence and uniqueness of weak solution

1 Mathematical model

Consider a two-dimensional fluid motion governed by the momentum and the continuity equation

$$\begin{aligned} \rho\partial_t \boldsymbol{v} + \rho\,(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} - \operatorname{div}\left[2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})\right] + \nabla p &= 0 \\ \operatorname{div}\boldsymbol{v} &= 0 \end{aligned} \qquad (1.1)$$

with ρ denoting the constant density of fluid, $\boldsymbol{v} = (v_1, v_2)$ the velocity vector, p the pressure, $e(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)$ the symmetric deformation tensor and μ the viscosity of the fluid. We assume that fluid is obeying the non-Newtonian shear-dependent model, cf. [14, Málek Nečas, Rokyta, Ružička], [25, Yeleswarapu], [13, Lukáčová, Zaušková]. A typical example is the following power-law model

$$\mu(e(\boldsymbol{v})) = \mu_0 + \mu_1(1 + |e(\boldsymbol{v})|^2)^{\frac{p-2}{2}}, \quad p > 1, \qquad (1.2)$$

see also Section 3.1 for a more general description of the considered non-Newtonian model. Note that according to the parameter p, the non-Newtonian fluid is either shear-thinning ($p < 2$) or shear thickening ($p \geq 2$). Models for fluids with the shear dependent viscosity are used in many areas of engineering science such as geophysics, glaciology, polymer mechanics, blood or food rheology. For $p > 2$ this model is an analogy of the so-called Ladyzhenskaja's fluid, for $p = 3$ it yields the Smagorinskij model of turbulence. In our recent article [13], where numerical simulations of blood flow has been presented, the shear-thinning model of Carreau has been used in order to model blood flow.

In this paper we will consider both the shear-thickening case, i.e., $p \geq 2$, as well as the shear thinning case $2 > p$. In particular, we will be able to show that for $p > (1+\sqrt{5})/2$ the weak solution of shear-dependent non-Newtonian fluids in a moving domain with a given domain deformation exists, see Theorem 4.1 for the existence of weak solution of the so-called regularized problem in the sense of Definition 2.1, and Theorem 6.2 for the existence result of original problem. Moreover, if $p \geq 2$ the uniqueness and continuous dependence on the given data can be shown for regularized problem, cf. Theorem 5.1. and Theorem 5.2. In all results for the regularized as well as original problem presented in this paper the moving computational domain is dependent on a given deformation function h. In fact, we obtain existence and uniqueness results for one iteration step with respect to the domain deformation. The convergence of iteration process can be proven in special cases and we will comment on it in Note 5.1.

We follow with the description of mathematical model. The computational domain

$$\Omega(\eta) \equiv \{(x_1, x_2, t);\ 0 < x_1 < L,\ 0 < x_2 < R_0(x_1) + \eta(x_1, t),\ 0 < t < T\}$$

is given by a reference radius function $R_0(x_1)$ and the unknown free boundary function $\eta(x_1, t)$ describing the domain deformation. The fluid and the geometry of the computational domain are coupled through the following Dirichlet boundary condition

$$v_2(x_1, R_0 + \eta, t) = \frac{\partial \eta}{\partial t}, \quad v_1(x_1, R_0 + \eta, t) = 0, \tag{1.3}$$

where $\Gamma_w = \{(x_1, x_2);\ x_2 = R_0(x_1) + \eta(x_1, t), x_1 \in (-L, L)\}$ is the deforming part of the boundary, $\boldsymbol{n}$ is the outward normal vector, $\boldsymbol{n} := (-\partial_{x_1}(R_0 + \eta), 1)$. Moreover, the normal component of the fluid stress tensor provides the forcing term for the coupled deformation equation of the free boundary η, that is modeled by the generalized string equation [20, 19, Quarteroni]

$$E\left[\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta - c\frac{\partial^3 \eta}{\partial t \partial x_1^2}\right](x_1, t) = \tag{1.4}$$
$$(-\mathbf{T}_f \boldsymbol{n} \cdot \boldsymbol{e}_2 - P_w \mathbf{I}\boldsymbol{n} \cdot \boldsymbol{e}_2) + aE\frac{\partial^2 R_0}{\partial x_1^2},$$

where $x_1 \in (-L, L)$, $\mathbf{T}_f = -p\mathbf{I} + 2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})$. Equation (1.4) is equipped with the following boundary and initial conditions

$$\eta(0, t) = \eta(L, t) = 0 \quad \text{and} \quad \eta(x_1, 0) = \frac{\partial \eta}{\partial t}(x_1, 0) = 0. \tag{1.5}$$

Positive constants $E,\ a,\ b,\ c$ appearing in (1.4) are given as follows, cf. [13],

$$E=\rho_w\hbar, \qquad a=\frac{|\sigma_z|}{\rho_w\left(1+\left(\frac{\partial R_0}{\partial x_1}\right)^2\right)^{3/2}},$$

$$b=\frac{\mathbf{T}_f\boldsymbol{n}\cdot\boldsymbol{e}_2+P_w\mathbf{I}\boldsymbol{n}\cdot\boldsymbol{e}_2}{R_0}+\frac{\mathcal{E}}{\rho_w(R_0+\eta)R_0}, \qquad c>0,$$

where $\mathcal{E}$ is the Young modulus, $\hbar$ the wall thickness, ρ_w the density of the vessel wall tissue, the coefficient $c=\gamma/(\rho_w\hbar)$, γ positive constant. $|\sigma_z|=G\kappa$ is the longitudinal stress, $\kappa=1$ is the Timoshenko's shear correction factor and G is the shear modulus, equal to $G=\mathcal{E}/2(1+\sigma)$ with $\sigma=1/2$ for incompressible materials. Note that the coefficients a, b are non-constant, however, according to the assumption (1.12) they are bounded from above and below. In what follows, for the sake of simplicity we work with constant coefficients $a,\ b$. It is easy to see that due to their boundedness, the analysis presented below will be analogous also in general case.

We complete the system (1.1) with the following boundary and initial conditions: on the inflow part of the boundary, which we shall denote Γ_{in}, we set

$$v_2(0,x_2,t)=0, \tag{1.6}$$

$$\left(2\mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_1}-p+P_{in}-\frac{\rho}{2}|v_1|^2\right)(0,x_2,t)=0 \tag{1.7}$$

for any $0<x_2<1$, $0<t<T$ and for a given function $P_{in}=P_{in}(x_2,t)$. On the opposite, outflow part of the boundary Γ_{out}, we set

$$v_2(L,x_2,t)=0\ , \tag{1.8}$$

$$\left(2\mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_1}-p+P_{out}-\frac{\rho}{2}|v_1|^2\right)(L,x_2,t)=0 \tag{1.9}$$

for any $0<x_2<1$, $0<t<T$ and for a given function $P_{out}=P_{out}(x_2,t)$. Finally, on the remaining part of the boundary, Γ_c, we set the flow symmetry condition

$$v_2(x_1,0,t)=0\ , \qquad \mu(|e(\boldsymbol{v})|)\frac{\partial v_1}{\partial x_2}(x_1,0,t)=0 \tag{1.10}$$

for any $0<x_1<L$, $0<t<T$. The initial conditions read

$$\boldsymbol{v}(x_1,x_2,0)=\mathbf{0} \quad \text{for any } 0<x_1<L,\ 0<x_2<h(x_1,0). \tag{1.11}$$

The problem (1.1)–(1.11) is also a generalization of the problem studied in [26], or [9], where the Newtonian flow was considered, see also [3, 4, 5, 6, 17, 10, 21, 22, 23] for other theoretical results on fluid-structure interaction problems or related problems. We will prove the existence and uniqueness of weak solution to (1.1)–(1.11) analogously as in works [26], [9], and thus additional techniques, in particular, theory of monotonous operators, will be used. In fact we prove the existence and uniqueness result in the case of moving domain, which deformation is given by a priori known function $h = R_0 + \eta^k$, $h \in W^{2,2}((0,T) \times (0,L)) \cap C^1([0,T] \times [0,L])$, $R_0(x_1) \in C^2([0,T] \times [0,L])$ satisfying

$$
\begin{aligned}
&0 < \alpha \leq h(x_1,t) \leq \alpha^{-1}\,, \quad 0 < \alpha \leq R_0(x_1,t) \leq \alpha^{-1} && (1.12)\\
&\left|\frac{\partial h(x_1,t)}{\partial x_1}\right| + \left|\frac{\partial h(x_1,t)}{\partial t}\right| \leq K && (1.13)\\
&h(0,t) = R_0(0), \quad h(L,t) = R_0(L), \quad h(x_1,0) = R_0(x_1) > 0.
\end{aligned}
$$

In a special case of $\lambda = 1$, cf. (1.16), (1.17), it is enough to assume that the domain deformation function $h \in W^{1,\infty}((0,T) \times (0,L))$.

Finally, the convergence for $\eta^{(k)} \to \eta,\ k \to \infty$ for special coefficients in deformation equation is proven in [9], we will address to this work at the end of this paper.

In numerical experiments we iterate with respect to the domain, cf. [9, 26]. This means, that in one iteration we look for a solution $(\boldsymbol{v}, p, \eta) = (\boldsymbol{v}^{(k+1)}, p^{(k+1)}, \eta^{(k+1)})$ of the following problem

$$
\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \operatorname{div}[2\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})] - \nabla p \quad \text{in} \quad \Omega(\eta^{(k)}), \tag{1.14}
$$

where $\eta^{(k)}(x_1,t) = \rho^{-1}(h(x_1,t) - R_0(x_1))$, and

$$
\begin{aligned}
&-E\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta - c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right](x_1,t) = -aE\frac{\partial^2 R_0}{\partial x_1^2}(x_1,t) + \qquad (1.15)\\
&\left[\mu(|e(\boldsymbol{v})|)\left\{-\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\frac{\partial \eta^{(k)}}{\partial x_1} + 2\frac{\partial v_2}{\partial x_2}\right\} - p + P_w\right](x_1, R_0 + \eta^{(k)}, t)
\end{aligned}
$$

for any $0 < x_1 < L$, $0 < t < T$, where we linearize term $\frac{E}{\rho_w(R_0+\eta)R_0}$ in b by $\frac{E}{\rho_w(R_0+\eta^{(k)})R_0}$.

Furthermore, in the analysis of problem (1.1)–(1.11) the boundary condition (1.3)-(1.4) is splitted analogously to [9] in following way

$$
\begin{aligned}
&\left[\mu(|e(\boldsymbol{v})|)\left\{\left(-\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\frac{\partial \eta^{(k)}}{\partial x_1} + 2\frac{\partial v_2}{\partial x_2}\right\} - p + P_w\right.\\
&\qquad\left. -\frac{\rho}{2}v_2\left(v_2 - \frac{\partial \eta^{(k)}}{\partial t}\right)\right](x_1, R_0(x_1) + \eta^{(k)}, t) \qquad (1.16)\\
&= \kappa\left(\lambda\frac{\partial \eta}{\partial t}(x_1,t) + (1-\lambda)\frac{\partial \eta^{(k)}}{\partial t}(x_1,t) - v_2(x_1, R_0(x_1) + \eta^{(k)}, t)\right)
\end{aligned}
$$

and

$$-E\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta - c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right](x_1,t) + Ea\frac{\partial^2 R_0}{\partial x_1^2}(x_1,t)$$
$$= \kappa\left(\lambda\frac{\partial\eta}{\partial t}(x_1,t) + (1-\lambda)\frac{\partial\eta^{(k)}}{\partial t}(x_1,t) - v_2(x_1, R_0(x_1)+\eta^{(k)}(x_1,t),t)\right) \tag{1.17}$$

with $\kappa \gg 1,\ 0 < \lambda \leq 1$.

We will show later, that this is a reasonable approximation. In fact, we prove the existence of solution if $\kappa \to \infty$ and thus we get the original boundary conditions (1.3) at the interface Γ_w.

Furthermore, we have to overcome the difficulties with solenoidal spaces. This will be done by means of the artificial compressibility, where we approximate the continuity equation with

$$\varepsilon\left(\frac{\partial p}{\partial t} - \Delta p\right) + \operatorname{div}\boldsymbol{v}_\varepsilon = 0,\ \varepsilon > 0.$$

By letting $\varepsilon \to 0$ we show that $\boldsymbol{v}_\varepsilon \to \boldsymbol{v}$, where $\boldsymbol{v}$ is the weak solution of (1.1). This is done analogous to [9]. Because of the lack of solenoidal property for velocity, we have the additional term in momentum equation $(1.1)_1$ $\frac{\rho}{2}v_i \operatorname{div}\boldsymbol{v}$, see also [24, Temam].

2 Weak formulation

In this section our aim is to present the weak formulation of the problem (1.1)–(1.11), which is reformulated on a fixed rectangular domain. Set

$$\begin{aligned} \boldsymbol{u}(y_1,y_2,t) &\stackrel{\text{def}}{=} \boldsymbol{v}(y_1, h(y_1,t)y_2, t) \\ q(y_1,y_2,t) &\stackrel{\text{def}}{=} \rho^{-1}p(y_1,h(y_1,t)y_2,t) \\ u(y_1,t) &\stackrel{\text{def}}{=} \frac{\partial\eta}{\partial t}(y_1,t) \end{aligned} \tag{2.1}$$

for $y \in D = \{(y_1,y_2);\ 0 < y_1 < L,\ 0 < y_2 < 1\}$, $0 < t < T$.
We define the the following space

$$\begin{aligned} \boldsymbol{V} &\equiv \left\{\boldsymbol{w} \in W^{1,p}(D):\ w_1 = 0 \ \text{ on } S_w \text{ and } w_2 = 0 \ \text{ on } S_{in}\cup S_{out}\cup S_c\right\}, \\ S_w &= \{(y_1,1) : 0 < y_1 < L\}, \\ S_{in} &= \{(0,y_2) : 0 < y_2 < 1\}, \\ S_{out} &= \{(L,y_2) : 0 < y_2 < 1\}, \\ S_c &= \{(y_1,0) : 0 < y_1 < L\}. \end{aligned} \tag{2.2}$$

Let us introduce the following notations

$$\operatorname{div}_h \boldsymbol{u} \stackrel{\text{def}}{=} \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} + \frac{1}{h}\frac{\partial u_2}{\partial y_2},$$

$$\begin{aligned} a_1(q,\phi) = \int_D \Bigg\{ & \left[h\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_1} \\ & + \left[\frac{1}{h}\frac{\partial q}{\partial y_2} - y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_2}\Bigg\}\, dy, \end{aligned} \tag{2.3}$$

$$((\boldsymbol{u},\boldsymbol{\psi})) = \int_D \tau_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}_{ij}(\boldsymbol{\psi}), \quad \tau_{ij}(\hat{e}(\boldsymbol{u})) = \rho^{-1}\mu(|\hat{e}(\boldsymbol{u})|)\hat{e}_{ij}(\boldsymbol{u}), \tag{2.4}$$

$$\hat{e}_{ij} = \sum_{i,j=1}^{2} \frac{1}{2}(\hat{\partial}_i(u_j) + \hat{\partial}_j(u_i)),$$

$$\hat{\partial}_1 = \sqrt{h}\left(\frac{\partial}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial}{\partial y_2}\right), \quad \hat{\partial}_2 = \frac{\partial}{\partial y_2}\frac{1}{\sqrt{h}},$$

$$\begin{aligned} b(\boldsymbol{w},\boldsymbol{m},\boldsymbol{\varpi}) \;=\; & \int_D \Bigg\{ \left(h u_1 \left(\frac{\partial \boldsymbol{z}^i}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{z}}{\partial y_2}\right) + u_2 \frac{\partial \boldsymbol{z}^i}{\partial y_2}\right) \cdot \boldsymbol{\psi} \\ & + \frac{h}{2}\, \boldsymbol{u}\boldsymbol{\psi} \operatorname{div}_h \boldsymbol{z} \Bigg\}\, dy \\ & - \frac{1}{2}\int_0^1 R_0 u_1 z_1 \boldsymbol{\psi}_1(L, y_2)\, dy_2 + \frac{1}{2}\int_0^1 R_0 u_1 z_1 \boldsymbol{\psi}_1(0, y_2)\, dy_2 \\ & - \frac{1}{2}\int_0^L u_2 z_2 \boldsymbol{\psi}_2(y_1, 1)\, dy_1 \end{aligned} \tag{2.5}$$

for $\boldsymbol{m} = (\boldsymbol{z}, \tilde{q}, \tilde{u})$, $\boldsymbol{w} = (\boldsymbol{u}, q, u)$, $\boldsymbol{\varpi} = (\boldsymbol{\psi}, v, \vartheta)$.

Definition 2.1 [Weak solution]
Let $\boldsymbol{u} \in L^p(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D))$, $q \in L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))$ and $u \in L^\infty(0,T;H_0^1(0,L)) \cap H^1(0,T;L^2(0,L))$. A triple $(\boldsymbol{u}, q, u)$ is called a weak solution of the regularized problem (1.1)–(1.11) if the following equation holds

$$
\begin{aligned}
&-\int_0^T \left\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi} \right\rangle dt \;= \\
&\int_0^T \int_D \left(-\frac{\partial h}{\partial t}\frac{\partial(y_2\boldsymbol{u})}{\partial y_2}\boldsymbol{\psi} + b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi}) - h\, q\, \mathrm{div}_h\boldsymbol{\psi} \right) dy \;+ ((\boldsymbol{u},\boldsymbol{\psi}))\, dt \\
&+\int_0^T \int_0^1 h(L,t)q_{out}\psi_1\; (L,y_2,t) - h(0,t)q_{in}\psi_1\; (0,y_2,t)\; dy_2 dt \\
&+\int_0^T \int_0^L \left(q_w + \frac{1}{2}u_2\frac{\partial h}{\partial t} + \frac{\kappa}{\rho}\left(u_2 - \lambda u - (1-\lambda)\frac{\partial h}{\partial t} \right) \right) \psi_2\; (y_1,1,t)\, dy_1 dt \\
&+\varepsilon \int_0^T \left\langle \frac{\partial(hq)}{\partial t}, \phi \right\rangle dt \qquad\qquad (2.6) \\
&+\int_0^T \int_D \left(-\varepsilon\frac{\partial h}{\partial t}\frac{\partial(y_2 q)}{\partial y_2}\phi + \varepsilon a_1(q,\phi) + \; h\, \mathrm{div}_h\boldsymbol{u}\; \phi \right) dy\, dt \\
&+\frac{\varepsilon}{2}\int_0^T \int_0^L \frac{\partial h}{\partial t}(y_1,t)\; q\phi\; (y_1,1,t)\, dy_1 dt \\
&+\int_0^T \int_0^L \left(\frac{\partial u}{\partial t}\xi + c\,\frac{\partial u}{\partial y_1}\frac{\partial \xi}{\partial y_1} + a\frac{\partial}{\partial y_1}\int_0^t u(y_1,s)ds\frac{\partial \xi}{\partial y_1} + b\int_0^t u(y_1,s)ds\; \xi \right. \\
&\qquad \left. - a\frac{\partial^2 R_0}{\partial y_1^2}\xi + \frac{\kappa}{E}\left(\lambda u + (1-\lambda)\frac{\partial h}{\partial t} - u_2 \right)\xi \right) (y_1,t)\, dy_1 dt
\end{aligned}
$$

for every
$(\boldsymbol{\psi},\phi,\xi) \in H_0^1(0,T;\boldsymbol{V}) \times L^2(0,T;H^1(D)) \times L^2(0,T;H_0^1(0,L))$.

3 Existence of weak solution

3.1 Preliminary properties for the shear-dependent model

Let us first specify more precisely the shear-dependent fluids that will be considered in this paper. We assume that there exists a potential $\mathcal{U} \in C^2(\mathbb{R}^{2\times 2})$, of shear stress tensor τ, such that for some $1 < p < \infty,\ C_1,\ C_2 > 0$ we have for all $\eta,\ \xi \in \mathbb{R}^{2\times 2}_{sym}$ and $i,\ j,\ k,\ l = 1,2$, cf. [14]

$$
\begin{aligned}
\frac{\partial \mathcal{U}(\eta)}{\partial \eta_{ij}} &= \tau_{ij}(\eta) && (3.1)\\
\mathcal{U}(\mathbf{0}) = \frac{\partial \mathcal{U}(\mathbf{0})}{\partial \eta_{ij}} &= 0 && (3.2)\\
\frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{mn}\partial \eta_{rs}}\xi_{mn}\xi_{rs} &\geq C_1\; (1+|\eta|)^{p-2}|\xi|^2 && (3.3)\\
\left| \frac{\partial^2 \mathcal{U}(\eta)}{\partial \eta_{ij}\partial \eta_{kl}} \right| &\leq C_2(1+|\eta|)^{p-2}. && (3.4)
\end{aligned}
$$

Note, that the stress tensor $\tau_{ij} = 2\mu(\hat{e}(\boldsymbol{u}))\hat{e}_{ij}(\boldsymbol{u}),\ i,j = 1,2$ with $\mu(\hat{e}(\boldsymbol{u}))$ defined in (1.2) satisfies (3.1)–(3.4).

In what follows we show some suitable properties, that will be needed in order to obtain a priori estimates.

Lemma 3.1 (Interpolation inequalities)**.**
Let φ be any function in $H^1(D)$ such that $\varphi = 0$ on S_w or S_c. Then there exists a constant $C = C(p,\theta)$ such that

$$\|\varphi\|_p \le c\|\nabla\varphi\|_2^{\theta}\|\varphi\|_2^{1-\theta} \quad \textit{for} \quad \frac{p-2}{p} \le \theta \le 1, \quad p \ge 2, \tag{3.5}$$

$$\|\varphi\|_{\frac{2p}{3p-4}} \le c\|\nabla\varphi\|_p^{\theta}\|\varphi\|_2^{1-\theta} \quad \textit{for} \quad \frac{2-p}{p-1} \le \theta \le 1, \quad p \le 2. \tag{3.6}$$

Proof: See the Nirenberg-Gagliago inequality [12, Henry].

Lemma 3.2.
Let φ be any function in $W^{1,p}(D)$ such that $\varphi = 0$ on S_w or S_c. Then for any $r \ge 1$ we have

$$\|\varphi\|_{L^r(S)} \le c\|\nabla\varphi\|_{L^p(D)}^{\frac{1}{r}}\|\varphi\|_{L^{\frac{p(r-1)}{p-1}}(D)}^{1-\frac{1}{r}} + c_1\|\varphi\|_{L^r(D)}, \tag{3.7}$$

in case $p \ge 2$ we have also

$$\|\varphi\|_{L^r(S)} \le c\|\nabla\varphi\|_{L^2(D)}^{\frac{1}{r}}\|\varphi\|_{L^2(D)}^{1-\frac{1}{r}} \quad \textit{for } r \ge 2. \tag{3.8}$$

Proof: Analogous to the proof of Proposition 3.2 in [26].
The following useful lemma has been obtained in [26], see also [9].

Lemma 3.3 (Ellipticity of the form $a_1(\cdot,\cdot)$)**.**
Let the assumptions (1.12) on $h(x_1,t)$ be satisfied. Then

$$a_1(v,v) \ge \frac{\alpha}{2+K^2}\int_D |\nabla v|^2\,dy \tag{3.9}$$

for any $v \in H^1(D)^2$, where $K = \max_{[0,T]\times[0,L]}\big(|\partial_{x_1}h|, |\partial_t h|\big)$

Lemma 3.4 (Coercitivity of the form $((\cdot,\cdot))$)**.**
The viscosity form defined in (2.4) satisfies for any $1 \le p < \infty$ the following

estimates: there exists $\delta > 0$ *such that*

$$1a)\ ((\boldsymbol{u},\boldsymbol{u})) \geq \delta\|\boldsymbol{u}\|_{1,p}^{p} + \delta\|\boldsymbol{u}\|_{1,2}^{2} \quad for \quad p \geq 2,$$

$$1b)\ ((\boldsymbol{u},\boldsymbol{u})) \geq \delta \int_D \min\{|\boldsymbol{u}|^p, |\boldsymbol{u}|^2\} \qquad or$$

$$\geq \delta \int_D |\hat{e}|(|\hat{e}|^{p-1} - 1) \geq \delta C_4 \int_D |\hat{e}|^p - 1 \quad for \quad 1 \leq p < 2,$$

$$2a)\ ((\boldsymbol{u}^1, \boldsymbol{u}^1 - \boldsymbol{u}^2)) - ((\boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2)) \geq$$

$$\delta \int_D |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^2 + |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^p \quad for \quad p \geq 2,$$

$$2b) Assume \quad |\hat{e}(\boldsymbol{u}^1)|, |\hat{e}(\boldsymbol{u}^2)| \leq R, \quad then\ for \quad 1 < p < 2$$

$$((\boldsymbol{u}^1, \boldsymbol{u}^1 - \boldsymbol{u}^2)) - ((\boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2)) \geq \delta(R) \int_D |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^2,$$

$$3.\ ((\boldsymbol{u}^1, \boldsymbol{u}^1 - \boldsymbol{u}^2)) - ((\boldsymbol{u}^2, \boldsymbol{u}^1 - \boldsymbol{u}^2)) \geq 0.$$

Proof: Assertion 1a). We have for $p \geq 2$

$$((\boldsymbol{u},\boldsymbol{u})) = \int_D \tau_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}_{ij}(\boldsymbol{u}) = \int_D \int_0^1 \frac{d}{ds}\frac{\partial \mathcal{U}(s\hat{e}(\boldsymbol{u}))}{\partial \hat{e}_{ij}}\, ds\, \hat{e}_{ij}(\boldsymbol{u})$$

$$= \int_D \int_0^1 \frac{\partial^2 \mathcal{U}(s\hat{e}(\boldsymbol{u}))}{\partial \hat{e}_{ij}\partial \hat{e}_{kl}}\, ds\, \hat{e}_{kl}(\boldsymbol{u})\hat{e}_{ij}(\boldsymbol{u}) \overset{(3.3)}{\geq} C_1 \int_D \int_0^1 (1 + s|\hat{e}(\boldsymbol{u})|)^{p-2} ds\, |\hat{e}(\boldsymbol{u})|^2$$

$$\overset{(1+s|\hat{e}|)^{p-2} \geq \frac{1}{2}(1+(s|\hat{e}|)^{p-2})}{\geq} \frac{C_1}{2} \int_D \int_0^1 1 + (s|\hat{e}(\boldsymbol{u})|)^{p-2} ds\, |\hat{e}(\boldsymbol{u})|^2$$

$$= \frac{C_1}{2}\int_D |\hat{e}(\boldsymbol{u})|^2 + \frac{C_1}{2(p-1)} \int_D s^{p-1}|\hat{e}(\boldsymbol{u})|^p \Big|_{s=0}^{s=1} |\hat{e}(\boldsymbol{u})|$$

$$= \frac{C_1}{2}\int_D |\hat{e}(\boldsymbol{u})|^2 + \frac{1}{p-1}|\hat{e}(\boldsymbol{u})|^p.$$

The assertion of lemma follows from the generalized Korn inequality, proven in [16, Nečas], [18, Pompe]. Indeed, according to [18], $\hat{e}(\boldsymbol{u})$ could be written in the form:

$$\hat{e}(\boldsymbol{u}) = \nabla \boldsymbol{u} F(y) + (\nabla \boldsymbol{u} F(y))^T, \text{ where}$$

$$F(y) = \frac{1}{2}\begin{pmatrix} \sqrt{h(y_1,t)} & 0 \\ -\frac{y_2}{\sqrt{h(y_1,t)}}\frac{\partial h(y_1,t)}{\partial y_1} & \frac{1}{\sqrt{h(y_1,t)}} \end{pmatrix}.$$

Since $F : \bar{D} \mapsto M^{2\times 2}(\mathbb{R})$ is a continuous mapping, $\det F(y) = 1$ and $\boldsymbol{u} \in \boldsymbol{V}$ (vanishing on some open subset of ∂D), we get according to the main result of [18]

$$\int_D |\hat{e}(\boldsymbol{u})|^p \geq c \int_D |\nabla \boldsymbol{u}|^p. \tag{3.10}$$

We should point out that the proof of this generalization of Korn's inequality with variable coefficient in [18] could be performed also for $\boldsymbol{u}$ vanishing on S component-wisely, a.e. $u_1 = 0$ on S_w, $u_2 = 0$ on $S_{out} \cup S_c \cup S_{in}$.

Assertions 1b), 2a), 2b), 3. are proven in [14, Lemma 1.19]. Note that on the base of (3.10) we can get also the differences $\|\boldsymbol{u}^1-\boldsymbol{u}^2\|_{1,2}^2$, $\|\boldsymbol{u}^1-\boldsymbol{u}^2\|_{1,p}^p$ on the right hand sides of assertions 2a), 2b). ■

Lemma 3.5 (Boundedness of the form $((\cdot,\cdot))$).
Let $\boldsymbol{u},\ \boldsymbol{v}\in\boldsymbol{V}$ *then for* $1<p<\infty$

$$((\boldsymbol{u},\boldsymbol{v}))\le C\|\boldsymbol{u}\|_{1,p}^{p-1}\|\boldsymbol{v}\|_{1,p} \tag{3.11}$$

Proof: for $1<p<\infty$

$$\begin{aligned}
((\boldsymbol{u},\boldsymbol{v})) &= \int_D \tau_{ij}(\hat{e}(\boldsymbol{u}))\hat{e}(\boldsymbol{v}) = \int_D\int_0^1 \frac{d}{ds}\frac{\partial\mathcal{U}(s\hat{e}(\boldsymbol{u}))}{\partial\hat{e}_{ij}}\,ds\;\hat{e}_{ij}(\boldsymbol{v})\\
&= \int_D\int_0^1 \frac{\partial^2\mathcal{U}\,(s\hat{e}(\boldsymbol{u}))}{\partial\hat{e}_{ij}\partial\hat{e}_{kl}}\hat{e}_{kl}(\boldsymbol{u})\;ds\;\hat{e}_{ij}(\boldsymbol{v})\\
&\overset{(3.4)}{\le} C_2\int_D\int_0^1(1+s|\hat{e}(\boldsymbol{u})|)^{p-2}|\hat{e}(\boldsymbol{u})|\;ds|\hat{e}(\boldsymbol{v})|\\
&= \frac{C_2}{p-1}\int_D[(1+|\hat{e}(\boldsymbol{u})|)^{p-1}-1]|\hat{e}(\boldsymbol{v})| \le \frac{C_p}{p-1}\int_D|\hat{e}(\boldsymbol{u})|^{p-1}|\hat{e}(\boldsymbol{v})|\\
&\le c_p\left(\int_D|\hat{e}(\boldsymbol{u})|^{\frac{p-1}{1}\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\int_D|\hat{e}(\boldsymbol{u})|^p\right)^{\frac{1}{p}} \le c\|\boldsymbol{u}\|_{1,p}^{p-1}\|\boldsymbol{v}\|_{1,p}.
\end{aligned}$$

Lemma 3.6 (Continuity of the form $((\cdot,\cdot))$).
The viscosity form $((\cdot,\cdot))$, *cf. (2.4), is continuous, i.e.,*

$$((\boldsymbol{u}^1,\boldsymbol{v}))-((\boldsymbol{u}^2,\boldsymbol{v}))\le C_4\|\boldsymbol{u}^1-\boldsymbol{u}^2\|_{1,p}^{p-1}\|\boldsymbol{v}\|_{1,p}+C_5(\|\boldsymbol{u}^2\|_{1,p}^{p-2}+1)\|\boldsymbol{u}^1-\boldsymbol{u}^2\|_{1,p}\|\boldsymbol{v}\|_{1,p}$$

for any $2<p<\infty$,

$$((\boldsymbol{u}^1,\boldsymbol{v}))-((\boldsymbol{u}^2,\boldsymbol{v}))\le C_4\|\boldsymbol{u}^1-\boldsymbol{u}^2\|_{1,p}^{p-1}\|\boldsymbol{v}\|_{1,p}$$

for any $1<p\le 2$.

Proof: Let $p > 2$ then

$$
\begin{aligned}
&((\boldsymbol{u}^1,\boldsymbol{v})) - ((\boldsymbol{u}^2,\boldsymbol{v})) = \int_D \left[\tau_{ij}(\hat{e}(\boldsymbol{u}^1)) - \tau_{ij}(\hat{e}(\boldsymbol{u}^2))\right] \hat{e}(\boldsymbol{v}) \\
&= \int_D \int_0^1 \frac{d}{ds} \frac{\partial U\left(s\hat{e}(\boldsymbol{u}^1) + (1-s)\hat{e}(\boldsymbol{u}^2)\right)}{\partial \hat{e}_{ij}} \, ds \; \hat{e}_{ij}(\boldsymbol{v}) \\
&= \int_D \int_0^1 \frac{\partial^2 U\left(s\hat{e}(\boldsymbol{u}^1) + (1-s)\hat{e}(\boldsymbol{u}^2)\right)}{\partial \hat{e}_{ij}\partial \hat{e}_{kl}} \left(\hat{e}_{ij}(\boldsymbol{u}^1) - \hat{e}_{ij}(\boldsymbol{u}^2)\right) \, ds \; \hat{e}_{ij}(\boldsymbol{v}) \\
&\overset{(3.4)}{\leq} C_2 \int_D \int_0^1 (1 + |s\hat{e}(\boldsymbol{u}^1 - \boldsymbol{u}^2) + \hat{e}(\boldsymbol{u}^2)|)^{p-2} (\hat{e}_{ij}(\boldsymbol{u}^1) - \hat{e}_{ij}(\boldsymbol{u}^2)) \, ds |\hat{e}(\boldsymbol{v})| \\
&\leq C_3(p-2) \int_D \left(1 + |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^{p-2} + |\hat{e}(\boldsymbol{u}_2)|^{p-2}\right) |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)| \, |\hat{e}(\boldsymbol{v})| \\
&\leq c_4 \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,p} \|\boldsymbol{v}\|_{1,q} + C_4 \left(\int_D |\hat{e}(\boldsymbol{u}^1) - \hat{e}(\boldsymbol{u}^2)|^{\frac{(p-1)p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_D |\hat{e}(\boldsymbol{v})|^p\right)^{\frac{1}{p}} \\
&\quad + C_5 \left(\int_D |\hat{e}(\boldsymbol{u}^2)|^{\frac{(p-2)p}{p-2}}\right)^{\frac{p-2}{p}} \left(\int_D |\hat{e}(\boldsymbol{u}_1) - \hat{e}(\boldsymbol{u}^2)|^p\right)^{\frac{1}{p}} \left(\int_D |\hat{e}(\boldsymbol{v})|^p\right)^{\frac{1}{p}}.
\end{aligned}
$$

For $1 < p \leq 2$ we have that $p - 2 \leq 0$ and therefore

$$
\begin{aligned}
&\int_D \int_0^1 (1 + |s\hat{e}(\boldsymbol{u}^1 - \boldsymbol{u}^2) + \hat{e}(\boldsymbol{u}^2)|)^{p-2} . (\hat{e}_{ij}(\boldsymbol{u}^1) - \hat{e}_{ij}(\boldsymbol{u}^2)) \, ds |\hat{e}(\boldsymbol{v})| \\
&\leq \frac{1}{p-1} \int_D |s\hat{e}(\boldsymbol{u}^1 - \boldsymbol{u}^2) + \hat{e}(\boldsymbol{u}^2)|^{p-1} \Big|_0^1 |\hat{e}(\boldsymbol{v})| \\
&= C_4 \int_D (|\hat{e}(\boldsymbol{u}^1 - \boldsymbol{u}^2) + \hat{e}(\boldsymbol{u}^2)|^{p-1} - |\hat{e}(\boldsymbol{u}^2)|^{p-1}) |\hat{e}(\boldsymbol{v})| \\
&\leq \int_D |\hat{e}(\boldsymbol{u}^1 - \boldsymbol{u}^2)|^{p-1} |\hat{e}(\boldsymbol{v})|
\end{aligned}
$$

We have obtained in this case $((\boldsymbol{u}^1,\boldsymbol{v})) - ((\boldsymbol{u}^2,\boldsymbol{v})) \leq C_4 \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,p}^{p-1} \|\boldsymbol{v}\|_{1,p}$. ■

Note 3.1. *Note that following the technique used in the previous proof we obtain also*

$$
\begin{aligned}
&\int_D \left[\tau_{ij}(\hat{e}(\boldsymbol{u}^1)) - \tau_{ij}(\hat{e}(\boldsymbol{u}^2))\right] \phi \\
&\leq C_6 \left\{ \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,p}^{p-1} + (\|\boldsymbol{u}^2\|_{1,p}^{p-2} + 1) \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,p} \right\} \|\phi\|_p
\end{aligned}
$$

Lemma 3.7 (Nonlinear convective term $b(\boldsymbol{w},\boldsymbol{m},\boldsymbol{\zeta})$)**.**
For the trilinear form $b(\boldsymbol{w},\boldsymbol{m},\boldsymbol{\zeta})$, $\boldsymbol{w} = (\boldsymbol{u},\cdot,\cdot)$, $\boldsymbol{m} = (\boldsymbol{z},\cdot,\cdot)$, $\boldsymbol{\zeta} = (\boldsymbol{\psi},\cdot,\cdot)$ defined in (2.5) we have

$$
b(\boldsymbol{w},\boldsymbol{m},\boldsymbol{\zeta}) = \frac{1}{2} B(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) - \frac{1}{2} B(\boldsymbol{u},\boldsymbol{\psi},\boldsymbol{z}), \tag{3.12}
$$

where

$$B(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\psi}) \equiv \int_D \left(hu_1 \left(\frac{\partial \boldsymbol{z}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{z}}{\partial y_2} \right) + u_2 \frac{\partial \boldsymbol{z}}{\partial y_2} \right) \cdot \boldsymbol{\psi}\, dy.$$

Proof: The assertion of this lemma is obtained by per partes integration in term $\int_D \frac{h}{2}\, \boldsymbol{z}\cdot\boldsymbol{\psi}\, \mathrm{div}_h\, \boldsymbol{u}\, dy$, see also [9, 26]. ■

3.2 Stationary solution

In this section we show the existence of weak solution for one discrete time step. The problem (1.1)–(1.11) is approximated by a sequence of stationary problems obtained by the implicit time discretization. Thus let us approximate time derivatives by means of first order backward finite differences

$$\frac{\partial(h\boldsymbol{u})}{\partial t} \approx \frac{h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}}{\Delta t}, \frac{\partial(hq)}{\partial t} \approx \frac{h^i q^i - h^{i-1}q^{i-1}}{\Delta t}, \ \frac{\partial u}{\partial t} \approx \frac{u^i - u^{i-1}}{\Delta t},$$

where $\boldsymbol{u}^i$, q^i and u^i denote approximations of unknown $\boldsymbol{u}$, q and u at time instance $i\Delta t$. We replace $\int_0^t u(s)\, ds$ by $\sum_{k=1}^i u^k \Delta t$. Moreover, let us use in this section the following notation

$$h^i(y_1) = h(y_1, i\Delta t), \quad q^i_{in/out/w}(y_2) = \frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} q_{in/out/w}(y_2, s)\, ds.$$

We introduce the following space

$$V \equiv \boldsymbol{V} \times H^1(D) \times H^1_0(0,L). \tag{3.13}$$

As a result the following variational problem is obtained from (2.6)

$$\begin{aligned} &\text{Find } \boldsymbol{w}^i = (\boldsymbol{u}^i, q^i, u^i) \in V \text{ such that} \\ &a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) + b^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{\varpi}) = L^i(\boldsymbol{\varpi}) \qquad \forall \boldsymbol{\varpi} \in V, \end{aligned} \tag{3.14}$$

where $\boldsymbol{\varpi} = (\boldsymbol{\omega}, v, \vartheta)$. Further
1. $a^i(\cdot,\cdot):\ V\times V \mapsto \mathbb{R}$ is the following continuous form on V

$$\begin{aligned} a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) &= ((\boldsymbol{u}^i, \boldsymbol{\omega})) + \varepsilon a_1(q^i, v) + \frac{1}{\Delta t}\int_D h^i\,(\boldsymbol{u}^i\boldsymbol{\omega} + q^i\, v)\; dy \\ &+ \int_0^L \left((c + a\Delta t)\frac{\partial u^i}{\partial y_1}\frac{\partial \vartheta}{\partial y_1} + \left(\frac{1}{\Delta t} + b\Delta t\right) u^i\vartheta \right) dy_1 \\ &- \int_D \frac{h^i - h^{i-1}}{\Delta t}\,\frac{\partial (y_2 \boldsymbol{u}^i)}{\partial y_2}\boldsymbol{\omega}\, dy + \int_0^L \frac{1}{2}u^i_2\frac{h^i - h^{i-1}}{\Delta t}\omega_2\; (y_1, 1)\, dy_1 \\ &- \varepsilon \int_D \frac{h^i - h^{i-1}}{\Delta t}\,\frac{\partial (y_2 q^i)}{\partial y_2} v\; dy + \frac{\varepsilon}{2}\int_0^L \frac{h^i - h^{i-1}}{\Delta t}\; q^i\, v\, (y_1, 1)\, dy_1 \\ &+ \kappa \int_0^L (\lambda u^i - u^i_2)\left(\frac{\vartheta}{E} - \omega_2\right)(y_1)\, dy_1 \\ &+ \int_D \left(h^i\, v\, \mathrm{div}_{h^i}\, \boldsymbol{u}^i - h^i q^i\, \mathrm{div}_{h^i}\, \boldsymbol{\omega}\right) dy, \end{aligned}$$

see also (2.3) and (2.4). Lemma 3.6 proves the continuity of this form.
2. The trilinear form $b^i(\cdot,\cdot,\cdot) : V \times V \times V \to \mathbb{R}$ is defined by (2.5). Note that $b^i(\boldsymbol{m}^i, \boldsymbol{m}^i, \boldsymbol{m}^i) = 0$, see Lemma 3.7, (3.12).
3. Finally, $L^i(\cdot)$ is the linear functional on V,

$$\begin{aligned} L^i(\boldsymbol{\varpi}) \quad = \quad & \frac{1}{\Delta t}\int_D h^{i-1}\left(\boldsymbol{u}^{i-1}\boldsymbol{\omega} + \varepsilon\, q^{i-1}\, v\right)\, dy + \frac{1}{\Delta t}\int_0^L u^{i-1}\vartheta\, dy_1 \\ & + \int_0^1 R_0(y_1)\left(q^i_{in}\,\omega_1\,(0,y_2) - q^i_{out}\,\omega_1\,(L,y_2)\right)\, dy_2 \\ & + \int_0^L \left(-q^i_w\,\omega_2\,(y_1,1) - \sum_{k=1}^{i-1}\left(a\frac{\partial u^k}{\partial y_1}\frac{\partial \vartheta}{\partial y_1} + bu^k\vartheta\right)(y_1)\Delta t\right)\, dy_1 \\ & +\kappa(1-\lambda)\int_0^L \frac{h^i - h^{i-1}}{\Delta t}\left(\omega_2 - \frac{\vartheta}{E}\right)(y_1)dy_1 + \int_0^L a\frac{\partial^2 R_0}{\partial y_1^2}\vartheta(y_1)dy_1. \end{aligned}$$

3.2.1 Existence of finite-dimensional solution

The existence of stationary solution is the consequence of coercivity of the viscosity form $((\cdot,\cdot))$ and of $a_1(\cdot,\cdot)$, see Lemma 3.4, Lemma 3.3, the continuity of these forms, see Lemma 3.6, and of the following lemma.

Lemma 3.8. *Let Y be a finite-dimensional Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Let P be a continuous mapping from Y into itself, such that for a sufficiently large $\varrho > 0$,*

$$(P(\zeta),\zeta) \geq 0 \quad \forall \zeta \in Y \text{ such that } \|\zeta\| = \varrho. \tag{3.15}$$

Then there exists $\zeta \in Y$, $\|\zeta\| \leq \varrho$ such that $P(\zeta) = 0$.

Proof: see [24, Lemma 2.1.4, p. 164].

The proof of existence of the finite-dimensional solution to (3.14) is analogous to the proof given in [9] or [26, Theorem 4.1]. In our case the finite-dimensional Hilbert space $Y = \mathscr{V}^m = \text{span}\{\xi_1, \dots \xi_m\}$, $\xi_i \in C^1$ is equipped with the norm $\|\cdot\|_{1,2}$ and P is a continuous mapping from Y into itself given by

$$(P(\boldsymbol{\zeta}), \boldsymbol{z}) = a^i(\boldsymbol{\zeta}, \boldsymbol{z}) + b^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}, \boldsymbol{z}) - L^i(\boldsymbol{z}) \quad \forall \boldsymbol{z} \in Y.$$

From Lemma 3.6 it is easy to see, that the assumption of continuity of P is fulfilled. With the assistance of Lemmas 3.4 and 3.3 we obtain the property (3.15). Indeed, we have for $1 < p < 2$

$$\begin{aligned} (P(\boldsymbol{\zeta}), \boldsymbol{\zeta}) = a^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}) - L^i(\boldsymbol{\zeta}) \geq \delta \int_D |\hat{e}(\boldsymbol{u})|^p - |\hat{e}(\boldsymbol{u})| dy - c_L\|\boldsymbol{u}\|_{1,p} \\ \geq c\delta\|\boldsymbol{u}\|^p_{1,p} - (c\delta|D| + c_L)\|\boldsymbol{u}\|_{1,p} \quad (3.16) \end{aligned}$$

for $\boldsymbol{\zeta} = (\boldsymbol{u}, 0, 0)$, $\boldsymbol{u} \in W^{1,p}(D)$. Therefore $(P(\boldsymbol{\zeta}), \boldsymbol{\zeta}) \geq 0$ in (3.16) for $\|\boldsymbol{u}\|_{1,p} = \varrho \geq \left(\frac{c\delta|D|+c_L}{c\delta}\right)^{1/(p-1)}$. Since $\|\boldsymbol{u}\|_{1,p} \leq c_e\|\boldsymbol{u}\|_{1,2}$ we have $\|\boldsymbol{u}\|_{1,2} \geq \frac{\varrho}{c_e}$. For $p \geq 2$ we have

$$(P(\boldsymbol{\zeta}), \boldsymbol{\zeta}) \geq \delta \int_D |\hat{e}(\boldsymbol{u})|^2 - c_L\|\boldsymbol{u}\|_{1,2} \geq c\delta\|\boldsymbol{u}\|_{1,2}^2 - c_L\|\boldsymbol{u}\|_{1,2}, \qquad (3.17)$$

therefore $(P(\boldsymbol{\zeta}), \boldsymbol{\zeta}) \geq 0$ for $\|\boldsymbol{u}\|_{1,2} = \frac{c_L}{c\delta}$ for $p \geq 2$. We have considered here only the terms obtaining the velocity vector $\boldsymbol{u}$, the estimates for other terms are straightforward, see [26].

Using Lemma 3.8 we obtain the existence of stationary weak solution to problem (3.14) which is bounded in $W^{1,2}(D)$, such that

$$\boldsymbol{w}^m = \sum_{k=1}^{m} c_k^m \xi_k \in \mathscr{V}^m, \quad \|\boldsymbol{w}^m\|_{1,2} \leq \varrho. \qquad (3.18)$$

To show the boundedness in $W^{1,p}$, we set test function $\boldsymbol{\varpi} = (\boldsymbol{u}^m, 0, 0)$ and get from (3.14)

$$a^i(\boldsymbol{w}^m, \boldsymbol{w}^m) = L^i(\boldsymbol{w}^m) \quad \forall \ \in \mathscr{V}^m, \qquad (3.19)$$

(written without temporal index i).
One can verify that for $\boldsymbol{w}^m = (\boldsymbol{u}^m, q^m, \lambda E u^m)$

$$\begin{aligned} a^i(\boldsymbol{w}^m, \boldsymbol{\varpi}) = ((\boldsymbol{u}^m, \boldsymbol{u}^m)) + \varepsilon a_1(q^m, q^m) + & \qquad (3.20) \\ \int_D \left[\frac{h^i}{\Delta t} + \frac{1}{2}\frac{h^i - h^{i-1}}{\Delta t}\right] \left(|\boldsymbol{u}^m|^2 + \varepsilon|q^m|^2\right) dy & \\ + \int_0^L \left((c + a\Delta t)\left|\frac{\partial u^m}{\partial y_1}\right|^2 + \left(\frac{1}{\Delta t} + b\Delta t\right)|u^m|^2\right) dy_1 & \end{aligned}$$

For sufficiently small $\Delta t = \alpha/K$, after omitting positive terms we get from Lemma 3.4

$$a^i(\boldsymbol{w}^m, \boldsymbol{w}^m) \geq c\delta\|\boldsymbol{u}^m\|_{1,p}^p \quad \text{for } p \geq 2,$$

$$a^i(\boldsymbol{w}^m, \boldsymbol{w}^m) \geq c\delta\|\boldsymbol{u}^m\|_{1,p}^p - c\delta|D|\|\boldsymbol{u}^m\|_{1,p} \quad \text{for } 1 < p < 2.$$

Since $|L^i(\boldsymbol{w}^m)| \leq c_L(\|\boldsymbol{u}^m\|_{1,p} + \|q^m\|_{1,2} + \|u^m\|_{1,2}) \leq c_L c\varrho$, see (3.18), we obtain from (3.19) the following estimates

$$\|\boldsymbol{u}^m\|_{1,p}^p \leq \frac{c_L c\varrho}{c\delta} \text{ for } p \geq 2, \qquad (3.21)$$

$$\|\boldsymbol{u}^m\|_{1,p}^p \leq \frac{c\delta|D|\|\boldsymbol{u}^m\|_{1,p} + c_L c\varrho}{c\delta} \leq \frac{c\delta|D|\varrho + c_L c\varrho}{c\delta} \text{ for } 1 < p < 2.$$

In view of boundedness in the reflexive Banach space $W^{1,p}(D)$ and the compact imbedding arguments, see [24, Theorem 1.1]

$$\begin{aligned} W^{1,p}(D) &\Subset L^r(D), \quad r \geq 1 \quad \text{for } p \geq 2, \qquad (3.22) \\ W^{1,p}(D) &\Subset L^r(D), \quad 1 \leq r < \frac{2p}{2-p} \quad \text{for } 1 \leq p \leq 2 \end{aligned}$$

we get the following convergences

$$
\begin{gathered}
\boldsymbol{u}^m \to \boldsymbol{u} \text{ in } L^r(D), \quad 1 \le r < \frac{2p}{2-p}; \text{ for } 1 \le p < 2 \\
\boldsymbol{u}^m \to \boldsymbol{u} \text{ in } L^r(D), \quad r \ge 1; \text{ for } p \ge 2 \\
\boldsymbol{u}^m \rightharpoonup \boldsymbol{u} \text{ in } W^{1,p}(D), \ p \ge 1 \\
q^m \to q \text{ in } L^r(D), \quad r \ge 1, \qquad q^m \rightharpoonup q \text{ in } W^{1,2}(D), \\
u^m \to u \text{ in } L^r(0,L), \quad r \ge 1, \qquad u^m \rightharpoonup u \text{ in } W^{1,2}(0,L).
\end{gathered}
\tag{3.23}
$$

3.2.2 Limiting process

We first explain the limiting process in the trilinear term $b(\cdot,\cdot,\cdot)$, (2.5). We show that $b(\boldsymbol{w}^m, \boldsymbol{w}^m, \xi^i) \to b(\boldsymbol{w}, \boldsymbol{w}, \xi^i)$, $m \to \infty$, $\xi^i \in Y = \mathscr{V}^m = \text{span}\{\xi_1, \dots \xi_m\}$, $\xi^i \in C^1$. Then we choose $\xi = (\boldsymbol{\omega}, \cdot, .\cdot) \in V$ and the test functions $\xi^i = (\boldsymbol{\omega}^i, \cdot, \cdot)$ such that $\boldsymbol{\omega}^i \to \boldsymbol{\omega}$ strongly in $W^{1,p}(D)$ as $i \to \infty$. The limiting process $b(\boldsymbol{w}, \boldsymbol{w}, \xi^i) \to b(\boldsymbol{w}, \boldsymbol{w}, \xi)$ as $i \to \infty$ follows easily.

Since $b(\boldsymbol{w}, \boldsymbol{m}, \xi) = \frac{1}{2}B(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\omega}) - \frac{1}{2}B(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{z})$, $\boldsymbol{w} = (\boldsymbol{u}, \cdot, \cdot)$, $\boldsymbol{m} = (\boldsymbol{\zeta}, \cdot, \cdot)$, see Lemma 3.7, we have

$$
\begin{aligned}
b(\boldsymbol{w}^m, \boldsymbol{w}^m, \xi^i) - b(\boldsymbol{w}, \boldsymbol{w}, \xi^i) &= b(\boldsymbol{w}^m - \boldsymbol{w}, \boldsymbol{w}^m, \xi^i) - b(\boldsymbol{w}, \boldsymbol{w} - \boldsymbol{w}^m, \xi^i) \\
&= \frac{1}{2}B(\boldsymbol{u}^m - \boldsymbol{u}, \boldsymbol{u}^m, \boldsymbol{\omega}^i) - \frac{1}{2}B(\boldsymbol{u}^m - \boldsymbol{u}, \boldsymbol{\omega}^i, \boldsymbol{u}^m) \\
&\quad - \frac{1}{2}B(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^m, \boldsymbol{\omega}^i) + \frac{1}{2}B(\boldsymbol{u}, \boldsymbol{\omega}^i, \boldsymbol{u} - \boldsymbol{u}^m) \\
&= \int_D -\frac{h}{2}(u_1^m - u_1)\left(\frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{u}^m}{\partial y_2}\right)\boldsymbol{\omega}^i + \dots \\
&\quad + \frac{h}{2}(u_1^m - u_1)\left(\frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{\omega}^i}{\partial y_2}\right)\boldsymbol{u}^m \, dy + \dots \\
+ \int_D \frac{h}{2}u_1 \left(\frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial(\boldsymbol{u}^m - \boldsymbol{u})}{\partial y_2}\right)\boldsymbol{\omega}^i &- \frac{h}{2}u_1\left(\frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{\omega}^i}{\partial y_2}\right)(\boldsymbol{u}^m - \boldsymbol{u}) \, dy + \dots,
\end{aligned}
\tag{3.24}
$$

where we pointed out only particular terms from (2.5). We show that the difference in (3.24) tends to 0. Indeed (3.24) is bounded from above by the following terms

$$
\begin{aligned}
c\|\boldsymbol{\omega}^i\|_{1,\infty}\left|\frac{\partial h}{\partial y_1}\right| \|\boldsymbol{u}^m - \boldsymbol{u}\|_{\frac{p}{p-1}} &(\|\nabla \boldsymbol{u}^m\|_p + \|\boldsymbol{u}^m\|_p + \|\boldsymbol{u}\|_p) \\
&+ \int_D \frac{h}{2}u_1\left(\frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial(\boldsymbol{u}^m - \boldsymbol{u})}{\partial y_2}\right)\boldsymbol{\omega}^i.
\end{aligned}
$$

This tends to 0 for any $p \ge 4/3$ due to the strong convergences in $L^r(D)$, cf. (3.23), boundedness in $W^{1,p}(D)$, cf. (3.21), and due to the weak convergence of $\nabla \boldsymbol{u}^m$ in $L^p(D)$. The latter is used to show the limit in the last integral term in the above expression. Indeed, the test function $u_1 \boldsymbol{\omega}^i$ is automatically from $L^q(D), q = \frac{p}{p-1}$ for $p \geq 2$. For $1 \leq p < 2$ we have $\boldsymbol{u} \in L^{\frac{2p}{2-p}}(D)$, therefore $u_1 \boldsymbol{\omega}^i \in L^{\frac{p}{p-1}}(D)$ if $\frac{p}{p-1} \leq \frac{2p}{2-p}$, i.e. $p \geq 4/3$.

Next, we show that

$$\int_D ((\boldsymbol{u}^m, \boldsymbol{\omega})) \to \int_D ((\boldsymbol{u}, \boldsymbol{\omega})), \quad \boldsymbol{\omega} \in W^{1,p}(D). \tag{3.25}$$

Let $\boldsymbol{u} \in W^{1,p}(D)$. We define $A(\hat{e}(\boldsymbol{u})) \in (L^p(D))^*$, $\hat{e} = \hat{e}(\boldsymbol{u}) \in L^p(D)$ in the following way

$$\langle A(\hat{e}(\boldsymbol{u})), \hat{e}(\boldsymbol{u}) \rangle = \int_D \tau_{ij}(\hat{e}(\boldsymbol{u})) \hat{e}_{ij}(\boldsymbol{u}),$$

see (2.4). First, using Lemma 3.4, assumption 3 we get monotonicity of that operator $A(\hat{e}(\boldsymbol{u}))$, i.e.

$$\langle A(\hat{e}(\boldsymbol{u})) - A(\hat{e}(\boldsymbol{v})), \hat{e}(\boldsymbol{u}) - \hat{e}(\boldsymbol{v}) \rangle \geq 0.$$

Note, that from (3.21) and from Lemma 3.5 we have for all $\boldsymbol{v} \in W^{1,p}(D)$

$$\langle A(\hat{e}(\boldsymbol{u}^m)), \hat{e}(\boldsymbol{v}) \rangle \leq C,$$

which implies the weak convergence $A(\hat{e}(\boldsymbol{u}^m)) \rightharpoonup f$. Moreover, from the monotonicity of operator A we have

$$\begin{aligned}
&\liminf_{m \to \infty} \langle A(\hat{e}(\boldsymbol{u}^m)) - A(\hat{e}(\boldsymbol{u})), \hat{e}(\boldsymbol{u}^m) - \hat{e}(\boldsymbol{u}) \rangle = \\
&\liminf_{m \to \infty} \{ -\langle A(\hat{e}(\boldsymbol{u})), \hat{e}(\boldsymbol{u}^m) - \hat{e}(\boldsymbol{u}) \rangle - \langle A(\hat{e}(\boldsymbol{u}^m)), \hat{e}(\boldsymbol{u}) \rangle + \langle A(\hat{e}(\boldsymbol{u}^m)), \hat{e}(\boldsymbol{u}^m) \rangle \} \\
&\geq 0
\end{aligned}$$

and thus we have

$$\liminf_{m \to \infty} \langle A(\hat{e}(\boldsymbol{u}^m)), \hat{e}(\boldsymbol{u}^m) \rangle \geq \langle f, \hat{e}(\boldsymbol{u}) \rangle.$$

Now, applying the Minty-Browder theorem for monotone operators we obtain that $f = A(\hat{e}(\boldsymbol{u}^m))$. As a consequence $A(\hat{e}(\boldsymbol{u}^m)) \rightharpoonup A(\hat{e}(\boldsymbol{u}))$, which is (3.25).

Let us summarize the main result of this section in the following theorem.

Theorem 3.1 (Stationary solution)**.**
Let $i \in \{1, 2, \ldots, n\}$ and $\boldsymbol{w}^j \in V$ for $j \leq i-1$ be given. Assume (3.1)–(3.4) and (1.12), (1.13), i.e. there are non-negative constants α, K, independent on i, such that

$$0 < \alpha \leq h^i(y_1) \leq \alpha^{-1},$$

and

$$\left| \frac{\partial h^i}{\partial y_1}(y_1) \right| + \left| \frac{h^i(y_1) - h^{i-1}(y_1)}{\Delta t} \right| \leq K$$

for all $0 \leq y_1 \leq L$ and $i = 1, 2, \ldots n$. Moreover, assume that

$$q^i_{in}, q^i_{out} \in L^2(0,1), \; q^i_w \in L^2(0, L) \quad \textit{and} \quad \Delta t \leq \alpha/K.$$

Then the problem (3.14) has at least one solution.

4 Existence of unsteady solution

4.1 A priori estimates

In this section we derive suitable a priori estimates for the sequence of piecewise constant and piecewise linear approximations in time of weak solution. Since our ultimate goal is to put the parameter $\kappa \to \infty$, our aim is to obtain such estimates, that are independent on κ.

We first rewrite (2.6) for piecewise constant $\boldsymbol{u},\ q,\ u$, replace time derivative in (2.6) with backward difference and replace integration in time by sum. This yields

$$
\begin{aligned}
&\Delta t \sum_{i=1}^{r} \Bigg[\int_D \Bigg\{ \left(\frac{h^i \boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}}{\Delta t} - \frac{h^i - h^{i-1}}{\Delta t} \frac{\partial (y_2 \boldsymbol{u}^i)}{\partial y_2} \right) \boldsymbol{\omega} \\
&\qquad + \left(h^i u_1^i \left(\frac{\partial \boldsymbol{u}^i}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) + u_2^i \frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) \boldsymbol{\omega} + \frac{h^i}{2} \boldsymbol{u}^i \boldsymbol{\omega} \operatorname{div}_{h^i} \boldsymbol{u}^i \\
&\qquad + ((u^i, \boldsymbol{\omega})) - h^i q^i \operatorname{div}_{h^i} \boldsymbol{\omega} \Bigg\} \, dy \\
&+ \int_D \left\{ \varepsilon \left(\frac{h^i q^i - h^{i-1} q^{i-1}}{\Delta t} - \frac{h^i - h^{i-1}}{\Delta t} \frac{\partial (y_2 q^i)}{\partial y_2} \right) v + \varepsilon a_1(q^i, v) + \operatorname{div}_{h^i} \boldsymbol{u}^i v \right\} \\
&+ \int_0^1 h(L,t) \left(q_{out}^i - \frac{1}{2} \left| u_1^i \right|^2 \right) \omega_1 \ (L, y_2) \, dy_2 \qquad (4.1) \\
&- \int_0^1 h(0,t) \left(q_{in}^i - \frac{1}{2} \left| u_1^i \right|^2 \right) \omega_1 \ (0, y_2) \, dy_2 \\
&+ \int_0^L \left(q_w^i - \frac{1}{2} u_2^i \left(u_2^i - \frac{h^i - h^{i-1}}{\Delta t} \right) \right) \omega_2 \ (y_1, 1) \, dy_1 \\
&+ \int_0^L \left(\frac{\kappa}{\rho} \left(u_2^i - \lambda u^i - (1-\lambda) \frac{h^i - h^{i-1}}{\Delta t} \right) \omega_2 + \frac{\varepsilon}{2} \frac{h^i - h^{i-1}}{\Delta t} q^i v \right) (y_1, 1) \, dy_1 \\
&+ \int_0^L \Bigg\{ \frac{u^i - u^{i-1}}{\Delta t} \vartheta + c \frac{\partial u^i}{\partial y_1} \frac{\partial \vartheta}{\partial y_1} + a \Delta t \sum_{k=1}^{i} \frac{\partial u^k}{\partial y_1} \frac{\partial \vartheta}{\partial y_1} - a \frac{\partial^2 R_0^i}{\partial y_1^2} \vartheta \\
&\quad + b \Delta t \left(\sum_{k=1}^{i} u^k \right) \vartheta + \frac{\kappa}{E} \left(\lambda u^i + (1-\lambda) \frac{h^i - h^{i-1}}{\Delta t} - u_2^i \right) \vartheta \Bigg\} (y_1) \, dy_1 \Bigg] \ = \ 0
\end{aligned}
$$

for any $\boldsymbol{\varpi} = (\boldsymbol{\omega}, v, \vartheta) \in V$.

Then we test the above identity with $(\boldsymbol{u}^i,\ q^i, \frac{E}{\rho}(\lambda u^i + (1-\lambda)\Upsilon h^i))$, $\Upsilon h^i = \frac{h^i - h^{i-1}}{\Delta t}$, multiply with 2 and perform following discrete calculus analogously as in [26].

$$2\sum_{i=1}^{r}\int_D \left(h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}\right)\boldsymbol{u}^i dy \;=\; \int_D h^r\,|\boldsymbol{u}^r|^2\,dy$$
$$+\sum_{i=1}^{r}\int_D \left\{\frac{1}{h^i}\left|h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}\right|^2 + \frac{h^{i-1}}{h^i}\left(h^i - h^{i-1}\right)\left|\boldsymbol{u}^{i-1}\right|^2\right\}dy\,,$$
$$-\Delta t\sum_{i=1}^{r}2\int_D \Upsilon h^i\,\frac{\partial(y_2\boldsymbol{u}^i)}{\partial y_2}\boldsymbol{u}^i\,dy$$
$$= -\Delta t\sum_{i=1}^{r}\int_0^L \Upsilon h^i\,\left|u_2^i\right|^2(y_1,1)\,dy_1 - \Delta t\sum_{i=1}^{r}\int_D \Upsilon h^i\left|\boldsymbol{u}^i\right|^2dy\,, \tag{4.2}$$
$$2\sum_{i=1}^{r}\int_0^L\left(u^i - u^{i-1}\right)u^i\,dy_1 = \int_0^L|u^r|^2\,dy_1 + \sum_{i=1}^{r}\int_0^L\left|u^i - u^{i-1}\right|^2dy_1\;,$$
$$a\Delta t\sum_{i=1}^{r}\int_0^L\frac{\partial U^i}{\partial y_1}\frac{\partial u^i}{\partial y_1}dy_1 = \frac{a}{2}\int_0^L\left\{\left|\frac{\partial U^r}{\partial y_1}\right|^2dy_1 + \sum_{i=1}^{r}\left|\frac{\partial(U^i - U^{i-1})}{\partial y_1}\right|^2\right\}dy_1,$$
$$b\Delta t\sum_{i=1}^{r}\int_0^L U^iu^i\,dy_1 \;= \frac{b}{2}\int_0^L\left\{|U^r|^2\,dy_1 + \sum_{i=1}^{r}\left|U^i - U^{i-1}\right|^2\right\}dy_1.$$

where U^i denotes

$$U^0 \equiv 0, \qquad U^i \equiv \sum_{k=1}^{i}u^k\Delta t, \qquad \frac{U^i - U^{i-1}}{\Delta t} = u^i, \tag{4.3}$$

Using (4.2), coercitivity and ellipticity properties of the forms $((\cdot,\cdot))$ and $a_1(\cdot,\cdot)$ (Lemma 3.4, Lemma 3.3), the Hölder inequality and the boundary imbedding (3.7) we get

$$\int_D h^r(|\boldsymbol{u}^r|^2 + \varepsilon|q^r|^2)dy + \frac{\lambda E}{\rho}\int_0^L|u^r|^2dy_1 \tag{4.4}$$
$$+\Delta t\sum_{i=1}^{r}\int_D 2\delta|\nabla\boldsymbol{u}^i|^p + \frac{2\alpha\varepsilon}{2+K^2}|\nabla q^i|^2dy + \frac{2\lambda cE}{\rho}\Delta t\sum_{i=1}^{r}\int_0^L\left|\frac{\partial u^i}{\partial y_1}\right|^2dy_1$$
$$+\int_0^L\frac{\lambda aE}{\rho}\left|\frac{\partial U^r}{\partial y_1}\right|^2 + \frac{\lambda bE}{\rho}|U^r|^2 + \frac{2\kappa}{\rho}\Delta t\sum_{i=1}^{r}[\lambda u^i + (1-\lambda)\Upsilon h^i - u_2^i]^2\,dy_1$$
$$\leq \quad F_h(u) + \Delta t\sum_{i=1}^{r}H^i\int_D h^i(|\boldsymbol{u}^i|^2 + \varepsilon|q^i|^2)dy + C_1\Delta t\sum_{i=1}^{r}\|\nabla\boldsymbol{u}^i\|_p\|q^i_{\partial D}\|_2$$
$$+\Delta t\sum_{i=1}^{r}\int_0^L\frac{2aE}{\rho}\frac{\partial^2R_0^i}{\partial y_1^2}(\lambda u^i + (1-\lambda)\Upsilon h^i)dy_1,$$

where

$$H^i \equiv \max_{0\le y_1\le L}\left[\left(\frac{1}{h^i} - \frac{h^{i-1}}{(h^i)^2}\right)\left(\Upsilon h^i\right)(y_1)\right]_+\,,$$

$$\|q^i_{\partial D}\|_2 \equiv \|q^i_{in}\|_{L^2(0,1)} + \|q^i_{out}\|_{L^2(0,1)} + \|q^i_w\|_{L^2(0,L)},$$

and

$$F_h(u) = \frac{2E}{\rho}(1-\lambda)\int_0^L \sum_{i=1}^r \Upsilon u^i \Upsilon h^i + c\frac{\partial u^i}{\partial y_1}\frac{\partial \Upsilon h^i}{\partial y_1} + a\frac{\partial U^i}{\partial y_1}\frac{\partial \Upsilon h^i}{\partial y_1} + bU^i\Upsilon h^i \, \Delta t \, dy_1. \tag{4.5}$$

Note that if $\lambda = 1$ we have $F_h(u) = 0$. Constant C_1 arises from (3.7) and the compact imbeddings $W^{1,p}(D) \Subset L^2(D)$, $W^{1,p}(D) \Subset L^{p/(p-1)}(D)$, cf. (3.22).

Moreover, in the case $p \le 2$ we use $((\boldsymbol{u},\boldsymbol{u})) \ge \delta C_4 \int_D |\hat{e}|^p - 1\, dy$, see Lemma 3.4, and the constant term $2\delta C_4|D|$ appears on the right hand side of (4.4). In what follows, we would like to obtain a priori estimates for $1 \le p < \infty$. We continue with case $p \ge 2$, in the case $1 \le p < 2$ the additional term $2\delta C_4|D|$ appears on the right hand side.

First, we use the discrete per partes calculus in the terms in $F_h(u)$

$$\begin{aligned}
\sum_{i=1}^r \int_0^L (u^i - u^{i-1})\Upsilon h^i &= \int_0^L u^r \Upsilon h^r - \sum_{i=1}^r \int_0^L u^{i-1}(\Upsilon h^i - \Upsilon h^{i-1}),\\
b\sum_{i=1}^r \int_0^L U^i(h^i - h^{i-1}) &= -b\sum_{i=1}^r \int_0^L U^i h^i - U^{i-1}h^{i-1} - b\sum_{i=1}^r \int_0^L h^i u^{i-1}\\
&= -b\int_0^L h^r U^r - b\sum_{i=1}^r \int_0^L h^i u^{i-1}.
\end{aligned}$$

Further, we apply Young's inequality in the terms on the right hand side of (4.4) and in (4.5) with appropriate constants $\varepsilon, C(\varepsilon)$; for example

$$C_1 \sum_{i=1}^r \|\nabla \boldsymbol{u}^i\|_p \|q^i_{\partial D}\|_2 \Delta t \le \delta \Delta t \sum_{i=1}^r \|\nabla \boldsymbol{u}^i\|_p^p + C_1 \frac{p-1}{p}\left(\frac{C_1}{p\delta}\right)^{\frac{1}{p-1}} \|q^i_{\partial D}\|_2^q.$$

After some calculation this yields for $\lambda \neq 1$

$$\begin{aligned}
&\xi^r + \sum_{i=1}^r \int_D \delta|\nabla \boldsymbol{u}^i|^p + \frac{2\alpha\varepsilon}{2+K^2}|\nabla q^i|^2 dy \Delta t + \frac{\lambda c E}{2\rho}\sum_{i=1}^r \int_0^L \left|\frac{\partial u^i}{\partial y_1}\right|^2 dy_1 \Delta t\\
&+ \int_0^L \lambda \frac{aE}{2\rho}\left|\frac{\partial U^r}{\partial y_1}\right|^2 + \lambda\frac{bE}{2\rho}|U^r|^2 + 2\sum_{i=1}^r \frac{\kappa}{\rho}[\lambda u^i + (1-\lambda)\Upsilon h^i - u^i_2]^2 dy_1 \Delta t\\
&\le R\Delta t \sum_{i=1}^r \xi^i + \Delta t \sum_{i=1}^r f^i,
\end{aligned} \tag{4.6}$$

where

$$
\begin{aligned}
\xi^r &= \int_D h^r(|\boldsymbol{u}^r|^2+\varepsilon|q^r|^2)dy+\frac{\lambda E}{2\rho}\int_0^L |u^r|^2 dy_1,\\
f^i &= M\|q_{\partial D}\|_2^q+\frac{2E(1-\lambda)^2}{\lambda}\int_0^L \max_{0\le\tau\le t}\left\{a\left|\frac{\partial h^i}{\partial y_1}\right|^2+b|h^i|^2+4|\Upsilon h^i|^2\right\}dy_1\\
&\quad+\frac{4E(1-\lambda)^2}{\lambda c}\int_0^L C_1|\Upsilon^2 h^i|^2+c^2\left|\frac{\partial\Upsilon h^i}{\partial y_1}\right|^2+a^2\left|\frac{\partial h^i}{\partial y_1}\right|^2+b^2C_1|h^i|^2dy_1\\
&\quad+\frac{2C_1E\lambda}{c}\int_0^L a^2\left|\frac{\partial^2 R_0^i}{\partial y_1^2}\right|^2dy_1+aE(1-\lambda)\int_0^L|\Upsilon h^i|^2dy_1,
\end{aligned}
$$

with $\quad M=C_1\frac{p-1}{p}\left(\frac{C_1}{p\,\delta}\right)^{\frac{1}{p-1}}\quad$ and $\quad R=\max_i H^i.$

Applying the discrete form of Gronwall's inequality (see [7, Evans]) to (4.6); in fact to the following inequality

$$\xi^r\le R\Delta t\sum_{i=1}^r\xi^i+\Delta t\sum_{i=1}^r f^i$$

we obtain

$$\Delta t\sum_{i=1}^r\xi^i\le e^{Rt}t\Delta t\sum_{i=1}^r f^i,\quad t=r\Delta t. \tag{4.7}$$

The right hand side of inequality (4.6) can be estimated with use of (4.7) by

$$(1+Re^{RT}T)\Delta t\sum_{i=1}^n f^i,\qquad T=n\Delta t.$$

This is bounded since $h\in W^{2,2}((0,T)\times(0,L))\cap C^1([0,T]\times[0,L])$. Consequently we obtain the **first a priori estimate:**

I.

$$
\begin{aligned}
&\max_{1\le r\le n}\int_D h^r(|\boldsymbol{u}^r|^2+\varepsilon|q^r|^2)dy+\frac{\lambda E}{\rho}\int_0^L|u^r|^2dy_1 \qquad (4.8)\\
&+\Delta t\sum_{i=1}^n\int_D\delta|\nabla\boldsymbol{u}^i|^p+\frac{2\alpha\varepsilon}{2+K^2}|\nabla q^i|^2dy+\lambda\frac{cE}{\rho}\Delta t\sum_{i=1}^n\int_0^L\left|\frac{\partial u^i}{\partial y_1}\right|^2dy_1\\
&+\int_0^L\lambda\frac{aE}{2\rho}\left|\frac{\partial U^r}{\partial y_1}\right|^2+\lambda\frac{bE}{2\rho}|U^r|^2+2\frac{\kappa}{\rho}\Delta t\sum_{i=1}^n\left|\lambda u^i+(1-\lambda)\Upsilon h^i-u_2^i\right|^2dy_1\\
&+\sum_{i=1}^n\int_D\frac{1}{h^i}|h^i\boldsymbol{u}^i-h^{i-1}\boldsymbol{u}^{i-1}|^2+\varepsilon|h^iq^i-h^{i-1}q^{i-1}|^2dy\\
&+\int_0^L\frac{\lambda E}{\rho}|u^i-u^{i-1}|^2dy_1\le\tilde{M}\Delta t\sum_{i=1}^n f^i,
\end{aligned}
$$

where $\tilde{M} = (1+Re^{RT}T)$, $T = n\Delta t$ and f^i depends only on given data. Thus, the right hand side of inequality (4.8) depending on deformation function $h(y_1, t)$ and the boundary pressures q_{in}, q_{out}, q_w is bounded. Note that constant $\tilde{M}$ does not depend on κ. We will see later in Section 6 that the continuous analogy of this estimates will be useful to prove convergence of the approximate solution for $\varepsilon \to 0$ and $\kappa \to \infty$.

Now we are ready to show some suitable properties of time differences of the weak solution. We first show with the assistance of the first a priori estimate (4.8), analogously to the [26], that time difference of the domain deformation velocity is bounded in $L^2((0,T) \times D)$. Therefore we again consider (4.1) and test (4.1) with $\boldsymbol{\psi}^i = (\mathbf{0},\ 0,\ E\Upsilon u^i)$. This yields

$$\begin{aligned}\Delta t \sum_{i=1}^{n} \int_0^L E\left|\Upsilon u^i\right|^2 + E\,c\frac{\partial u^i}{\partial y_1}\frac{\partial \Upsilon u^i}{\partial y_1} + E\,a\left(\sum_{k=1}^{i}\frac{\partial u^k}{\partial y_1}\Delta t\right)\frac{\partial \Upsilon u^i}{\partial y_1}\\ +E\,b\left(\sum_{k=1}^{i} u^k\Delta t\right)\Upsilon u^i + \kappa(\lambda u^i + (1-\lambda)\Upsilon h^i - u_2)\Upsilon u^i \\ +E\,a\frac{\partial^2 R_0^i}{\partial y_1^2}\Upsilon u^i\ dy_1 = 0.\end{aligned} \tag{4.9}$$

Next, we use (4.2-c) in term $Ec\frac{\partial u^i}{\partial y_1}\frac{\partial \Upsilon u^i}{\partial y_1}$ of (4.9) and following discrete calculus in terms obtaining $E\,a$ and $E\,b$

$$\begin{aligned}\sum_{i=1}^{n}\int_0^L U^i(u^i - u^{i-1}) &= -\int_0^L U^r u^r + \Delta t\sum_{i=1}^{n}\int_0^L u^{i-1}\underbrace{\Delta t^{-1}\left(U^i - U^{i-1}\right)}_{u^i}\\ \sum_{i=1}^{n}\int_0^L \frac{\partial U^i}{\partial y_1}\frac{\partial(u^i - u^{i-1})}{\partial y_1} &= -\int_0^L \frac{\partial U^r}{\partial y_1}\frac{\partial u^r}{\partial y_1}\\ &\quad +\Delta t\sum_{i=1}^{n}\int_0^L \frac{\partial u^{i-1}}{\partial y_1}\underbrace{\Delta t^{-1}\frac{\partial\left(U^i - U^{i-1}\right)}{\partial y_1}}_{\frac{\partial u^i}{\partial y_1}}.\end{aligned} \tag{4.10}$$

Putting the terms with constants a, b, κ on the right hand side of (4.9) leads to

$$\begin{aligned}\Delta t\sum_{i=1}^{n}\int_0^L E\left|\Upsilon u^i\right|^2 + \frac{cE}{2}\int_0^L\left|\frac{\partial u^r}{\partial y_1}\right|^2 + \frac{cE}{2}\Delta t\sum_{i=1}^{n}\int_0^L\left|\frac{\partial(u^i - u^{i-1})}{\partial y_1}\right|^2 =\\ \int_0^L -bEU^r u^r - aE\frac{\partial U^r}{\partial y_1}\frac{\partial u^r}{\partial y_1} + \Delta t\sum_{i=1}^{n}\left(bEu^i u^{i-1} + aE\frac{\partial u^i}{\partial y_1}\frac{\partial u^{i-1}}{\partial y_1}\right.\\ \left. + \kappa(\lambda u^i + (1-\lambda)\Upsilon h^i - u_2)\Upsilon u^i + aE\frac{\partial^2 R_0^i}{\partial y_1^2}\Upsilon u^i\right) dy_1.\end{aligned} \tag{4.11}$$

Using Young's inequalities we estimate the right hand side of (4.11) with

$$\int_0^L \frac{bE}{2}\left(|U^r|^2+|u^r|^2\right)+\frac{cE}{4}\left|\frac{u^r}{\partial y_1}\right|^2+\frac{a^2E}{c}\left|\frac{U^r}{\partial y_1}\right|^2 dy_1 \tag{4.12}$$
$$+\Delta t\sum_{i=1}^n\int_0^L bE|u^i|^2+aE\left|\frac{u^i}{\partial y_1}\right|^2+\frac{\kappa^2}{E}(\lambda u^i+(1-\lambda)\Upsilon h^i-u_2)^2$$
$$+a^2E\left|\frac{\partial^2 R_0^i}{\partial y_1^2}\right|^2+\frac{E}{2}\left|\Upsilon u^i\right|^2 dy_1.$$

Subtracting appropriate terms and using the boundedness of the remaining terms in the right hand side of (4.12), cf. (4.8), we obtain analogously to [9, 26] the following **second a priori estimate**:

II a). (4.13)

$$\frac{E}{2}\sum_{i=1}^n\int_0^L\left|\frac{u^i-u^{i-1}}{\Delta t}\right|^2 dy_1\,\Delta t+\frac{cE}{4}\int_0^L\left|\frac{\partial u^r}{\partial y_1}\right|^2+\sum_{i=1}^n\left|\frac{\partial(u^i-u^{i-1})}{\partial y_1}\right|^2 dy_1$$
$$\le C\kappa\tilde{M}\sum_{i=1}^n f^i\Delta t.$$

Let us point out the reason for the linear dependence of the right hand side on κ. By Young's inequality we get on the right hand side the term

$$\kappa^2(\lambda u^i+(1-\lambda)\Upsilon h^i-u_2^i)^2.$$

Applying (4.8) we have

$$\sum_{i=1}^r\int_0^L\kappa^2[\lambda u^i+(1-\lambda)\Upsilon h^i-u_2^i]^2\Delta t dy_1\le\kappa\tilde{M}\sum_{i=1}^n f^i\Delta t.$$

The reader will see later, that the linear dependence in the second a priori estimate **II. a)** is not essential for the limiting process $\kappa\longrightarrow\infty$.

Let us define

$$\boldsymbol{U}^i=(h^i\boldsymbol{u}^i),\quad Q^i=h^iq^i.$$

Using the discrete sequences $\{\boldsymbol{U}^i\}_{i=1}^n$, $\{Q^i\}_{i=1}^n$, $\{u^i\}_{i=1}^n$ we construct on time interval $[0,T]$ in the common way the step functions

$$\boldsymbol{u}_n^s(y,t),\ q_n^s(y,t),\ \boldsymbol{U}_n^s(y,t),\ Q_n^s(y,t),\ u_n^s(y_1,t)$$

and the piecewise linear approximations of weak solution

$$\boldsymbol{u}_n(y,t),\ q_n(y,t),\ \boldsymbol{U}_n(y,t),\ Q_n(y,t),\ u_n(y_1,t).$$

We show now a priori estimate for the time derivative of piecewise linear approximation of weak solution. To this goal we test (2.6) with $(\boldsymbol{\psi},0,0)$, $\boldsymbol{\psi} \in L^{\alpha_1}(0,T;W^{1,p}(D))$, such that $\psi_1 = 0$ on S_w and $\psi_2 = 0$ on $S_{in} \cup S_{out} \cup S_c$ and α_1 will be specified later. From (2.6) we have

$$-\int_0^T \left\langle \frac{\partial \boldsymbol{U}_n}{\partial t}, \boldsymbol{\psi} \right\rangle dt = \dots \int_0^T \int_D ((\boldsymbol{u}_n, \boldsymbol{\psi})) + b(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}) \dots dy\, dt.$$

We concentrate only on particular terms that yields some restrictions. Estimates for other terms do not yield additional difficulties. According to Lemma 3.7 we have $2b(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}) = B(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}) - B(\boldsymbol{u}_n, \boldsymbol{\psi}, \boldsymbol{u}_n)$. Now, using the Hölder inequality, where $\frac{2(p-1)}{3p-4} \le a \le \frac{2(p-1)}{2-p}$ for $p \ge \frac{3}{2}$ we have

$$\begin{aligned} \int_0^T B(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}) &\le c \int_0^T \|\boldsymbol{u}_n\|_{1,p} \|\boldsymbol{u}_n\|_{\frac{ap}{(a-1)(p-1)}} \|\boldsymbol{\psi}\|_{\frac{ap}{p-1}} \qquad (4.14) \\ \left(\text{taking } a = \frac{2(p-1)}{2-p}\right) &\le c \int_0^T \|\boldsymbol{u}_n\|_{1,p} \|\boldsymbol{u}_n\|_{\frac{2p}{(3p-4)}} \|\boldsymbol{\psi}\|_{\frac{2p}{2-p}}. \end{aligned}$$

For $p \ge 2$ the space $L^2(D)$ is imbedded into $L^{\frac{2p}{(3p-4)}}(D)$ and we have from (4.14)

$$\begin{aligned} \int_0^T B(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}) &\le c \|\boldsymbol{u}_n\|_{L^\infty(0,T;L^2(D))} \int_0^T \|\boldsymbol{u}_n\|_{1,p} \|\boldsymbol{\psi}\|_{1,p} \\ &\le c \|\boldsymbol{u}_n\|_{L^\infty(0,T;L^2(D))} \|\boldsymbol{u}_n\|_{L^p(0,T;W^{1,p}(D))} \|\boldsymbol{\psi}\|_{L^{\frac{p}{p-1}}(0,T;W^{1,p}(D))} \le C \end{aligned}$$

for $\boldsymbol{\psi} \in L^p(0,T;W^{1,p}(D))$, since $\frac{p}{p-1} \le p$.

For $p < 2$ we use the Nirenberg-Gargliago inequality (3.6), cf. [12]. Setting $\theta = \frac{2-p}{p-1}$ and using the compact imbeddings, see (3.22), the right hand side of (4.14) can be upper bounded by

$$\int_0^T \|\boldsymbol{u}_n\|_{1,p} \|\boldsymbol{u}_n\|_{\frac{2p}{(3p-4)}} \|\boldsymbol{\psi}\|_{\frac{2p}{2-p}} \le c \|\boldsymbol{u}_n\|_{L^\infty(0,T;L^2(D))}^{\frac{2p-3}{p-1}} \int_0^T \|\boldsymbol{u}_n\|_{1,p}^{\frac{1}{p-1}} \|\boldsymbol{\psi}\|_{1,p}.$$

Using the Hölder inequality for $P = p(p-1)$, $Q = \frac{p^2-p}{p^2-p-1}, p \ge \frac{1+\sqrt{5}}{2}$, $\frac{1}{P} + \frac{1}{Q} = 1$ yields

$$\begin{aligned} &\int_0^T B(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}) \le \qquad (4.15) \\ &\quad c \|\boldsymbol{u}_n\|_{L^\infty(0,T;L^2(D))}^{\frac{2p-3}{p-1}} \|\boldsymbol{u}_n\|_{L^p(0,T;W^{1,p}(D))}^{\frac{1}{p-1}} \|\boldsymbol{\psi}\|_{L^{\frac{p^2-p}{p^2-p-1}}(0,T;W^{1,p}(D))}. \end{aligned}$$

Let us now estimate $\int_0^T B(\boldsymbol{u}_n, \boldsymbol{\psi}, \boldsymbol{u}_n)$. Using the Hölder inequality and the Nirenberg-Gargliago inequality $\|\varphi\|_{\frac{2p}{(p-1)}} \le c \|\varphi\|_{1,p}^\theta \|\varphi\|_2^{(1-\theta)}$, where $\frac{1}{2p-2} \le$

$\theta \leq 1,\ p \geq 3/2$ we obtain

$$\int_0^T B(\boldsymbol{u}_n, \boldsymbol{\psi}, \boldsymbol{u}_n) \leq c \int_0^T \|\boldsymbol{\psi}\|_{1,p} \|\boldsymbol{u}_n\|^2_{\frac{2p}{p-1}} \leq c \|\boldsymbol{u}_n\|^{\frac{2p-3}{p-1}}_{L^\infty(0,T;L^2(D))} \int_0^T \|\boldsymbol{\psi}\|_{1,p} \|\boldsymbol{u}_n\|^{\frac{1}{p-1}}_{1,p}.$$

Now, using the Hölder inequality we get for $p \geq \frac{1+\sqrt{5}}{2}$ analogously as above

$$\int_0^T B(\boldsymbol{u}_n, \boldsymbol{\psi}, \boldsymbol{u}_n) \leq \tag{4.16}$$
$$c \|\boldsymbol{u}_n\|^{\frac{2p-3}{p-1}}_{L^\infty(0,T;L^2(D))} \|\boldsymbol{u}_n\|^{\frac{1}{p-1}}_{L^p(0,T;W^{1,p}(D))} \|\boldsymbol{\psi}\|_{L^{\frac{p^2-p}{p^2-p-1}}(0,T;W^{1,p}(D))}.$$

Note that for $p \geq 2$ we have $\frac{p^2-p}{p^2-p-1} \leq p$. We summarize the above considerations in the following way

$$\int_0^T b(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}) \leq C \text{ for } \begin{cases} p = \frac{1+\sqrt{5}}{2},\ \boldsymbol{\psi} \in L^{\alpha_1}(0,T;W^{1,p}(D)),\ \alpha_1 = \infty, \\ p \in (\frac{1+\sqrt{5}}{2}, 2),\ \boldsymbol{\psi} \in L^{\alpha_1}(0,T;W^{1,p}(D)), \\ \qquad \alpha_1 = \frac{p^2-p}{p^2-p-1}, \\ p \in \langle 2, \infty),\ \boldsymbol{\psi} \in L^{\alpha_1}(0,T;W^{1,p}(D)),\ \alpha_1 = p. \end{cases} \tag{4.17}$$

Further using Lemma 3.5 for $1 < p \leq \infty$ we get

$$\begin{aligned} \int_0^T ((\boldsymbol{u}_n, \boldsymbol{\psi})) &\leq c \int_0^T \|\boldsymbol{\psi}\|_{1,p} \|\boldsymbol{u}_n\|^{p-1}_{1,p} \\ &\leq c \|\boldsymbol{\psi}\|_{L^p(0,T;W^{1,p}(D))} \|\boldsymbol{u}_n\|^{p-1}_{L^p(0,T;W^{1,p}(D))}. \end{aligned} \tag{4.18}$$

Therefore the viscous term $\int_0^T ((\boldsymbol{u}_n, \boldsymbol{\psi}))$ is bounded. Consequently we have proved that

$$\int_0^T \left\langle \frac{\partial \boldsymbol{U}_n}{\partial t}, \boldsymbol{\psi} \right\rangle dt \leq C \quad \forall \boldsymbol{\psi} \in L^{\alpha_1}(0,T;W^{1,p}(D)),\ p \in \left\langle \frac{1+\sqrt{5}}{2}, \infty \right),$$

where α_1 is given by (4.17). The **second a priori estimate** for velocity reads

II b)

$$\frac{\partial \boldsymbol{U}_n}{\partial t} \in L^q(0,T;(W^{1,p}(D))^*) \text{ for } p \in \left\langle \frac{1+\sqrt{5}}{2}, \infty \right), \tag{4.19}$$

where q is given by $\frac{1}{q} + \frac{1}{\alpha_1} = 1, \alpha_1$ is defined by (4.17).

Note 4.1. *If we take more regular test functions $\boldsymbol{\psi} \in L^{\alpha_1}(0,T;W^{2,2}(D) \cap W^{1,p}(D))$, $\alpha_1 = \frac{2p-2}{2p-3}$ being specified by the appropriate Hölder inequalities, we can bound the trilinear term for $p \geq 3/2$*

$$\int_0^T b(\boldsymbol{u}_n, \boldsymbol{u}_n, \boldsymbol{\psi}) \leq C \|\boldsymbol{u}\|^{\frac{p}{2p-2}}_{L^p(0,T;W^{1,p}(D))} \|\boldsymbol{\psi}\|_{L^{\frac{2p-2}{2p-3}}(0,T;W^{2,2}(D))}.$$

In this case (4.19) is valid for $p \in \left\langle \frac{3}{2}, \infty \right)$ and q such that $\frac{1}{q} + \frac{2p-3}{2p-2} = 1$.

By testing (2.6) with $(0,\phi,0)$ we can also show that $\int_0^T \langle\sqrt{\varepsilon}Q_n,\phi\rangle_{H^1} \leq C\left(\frac{1}{\sqrt{\varepsilon}}\right)$ i.e., the **second a priori estimate** for pressure

II c)

$$\frac{\sqrt{\varepsilon}\partial Q_n}{\partial t} \in L^2(0,T;(H^1(D))^*). \tag{4.20}$$

Let us summarize the above results in the following lemma.

Lemma 4.1 (A priori estimates)**.**
Let us assume that $h \in W^{2,2}((0,T)\times(0,L)) \cap C^1([0,T]\times[0,L])$, and the assumptions (1.12), (3.1)–(3.4) hold. Then we have for approximate sequences of piecewise constant and piecewise linear functions the following results

$$\{\boldsymbol{u}_n^s\}_{n=0}^\infty,\ \{\boldsymbol{U}_n^s\}_{n=0}^\infty,\ \{\boldsymbol{U}_n\}_{n=0}^\infty\ \in L^p(0,T;\boldsymbol{V})\cap L^\infty(0,T;L^2(D)), \tag{4.21}$$

$$\left\{\frac{\partial \boldsymbol{U}_n}{\partial t}\right\}_{n=0}^\infty \in \begin{cases} L^{\frac{p}{p-1}}(0,T;(W^{1,p})^*) \ \text{for}\ p\in\langle 2,\infty), \\ L^1(0,T;(W^{1,p})^*) \ \text{for}\ p=\frac{1+\sqrt{5}}{2}, \\ L^{p'}(0,T;(W^{1,p})^*) \ \text{for}\ p\in\left(\frac{1+\sqrt{5}}{2},2\right), \\ \quad \frac{1}{p'}+\frac{p^2-p-1}{p^2-p}=1, \end{cases} \tag{4.22}$$

$$\{q_n^s\}_{n=0}^\infty,\{Q_n^s\}_{n=0}^\infty,\{Q_n\}_{n=0}^\infty \in L^2(0,T;W^{1,2}(D))\cap L^\infty(0,T;L^2(D)), \tag{4.23}$$

$$\left\{\sqrt{\varepsilon}\frac{\partial Q_n}{\partial t}\right\}_{n=0}^\infty \in L^2(0,T;H^{-1}(D)), \tag{4.24}$$

$$\{u_n^s\}_{n=0}^\infty,\ \{u_n\}_{n=0}^\infty\ \in L^2(0,T;H_0^1(0,L))\cap L^\infty(0,T;H^1(0,L)), \tag{4.25}$$

$$\left\{\frac{\partial u_n}{\partial t}\right\}_{n=0}^\infty \in L^2(0,T;(L^2(0,L)). \tag{4.26}$$

Proof: These results follow from a priori estimates (4.8), (4.13), (4.19), (4.20). ■

Finally, as a consequence of the above lemma we obtain the following results

Lemma 4.2.
Let $p>\frac{1+\sqrt{5}}{2}$. Then there exists a subsequence of $\{h_n^s,h_n,\boldsymbol{u}_n^s,\boldsymbol{u}_n,\boldsymbol{U}_n^s,\boldsymbol{U}_n,q_n^s,q_n,u_n^s,u_n\}_{n=1}^\infty$ and functions $\boldsymbol{u}\in L^p(0,T;\boldsymbol{V})\cap L^\infty(0,T;L^2(D))$, $q\in L^2(0,T;H^1(D))\cap L^\infty(0,T;L^2(D))$ and $u\in L^2(0,T;H_0^1(0,L))\cap L^\infty(0,T;L^2(0,L))$ (we denote the subsequence for simplicity again by $\{h_n^s,h_n,\boldsymbol{u}_n^s,\boldsymbol{u}_n,\boldsymbol{U}_n^s,\boldsymbol{U}_n,q_n^s,q_n,u_n^s,u_n\}_{n=1}^\infty$), such that

$$h_n \longrightarrow h \qquad \textit{strongly in}\ \ W^{1,\infty}(0,T;C[0,L]), \tag{4.27}$$

$$h_n^s \longrightarrow h \qquad \textit{strongly in}\ \ L^\infty(0,T;C^1[0,L]), \tag{4.28}$$

$$\boldsymbol{U}_n,\ \boldsymbol{U}_n^s \rightharpoonup h\boldsymbol{u}\ ,\boldsymbol{u}_n^s \rightharpoonup \boldsymbol{u} \textit{ weakly in } L^p(0,T;\boldsymbol{V}),$$

$$\left.\begin{array}{rl} \boldsymbol{U}_n,\ \boldsymbol{U}_n^s \longrightarrow & h\boldsymbol{u} \\ \boldsymbol{u}_n,\ \boldsymbol{u}_n^s \longrightarrow & \boldsymbol{u} \end{array}\right\} \begin{array}{ll} \textit{strongly in } L^r((0,T)\times D), & 1\le r<7/3, \\ \textit{strongly in } L^2(0,T;L^r(S)), & r>1, \end{array} \tag{4.29}$$

$$\left.\begin{array}{rl} \boldsymbol{U}_n^s,\ \boldsymbol{U}_n \longrightarrow & h\boldsymbol{u} \\ \boldsymbol{u}_n^s,\ \boldsymbol{u}_n \longrightarrow & \boldsymbol{u} \end{array}\right\} \begin{array}{l} \textit{strongly in } L^p(0,T;L^p(D)), \\ p>1, \end{array} \tag{4.30}$$

$$\begin{array}{rll} q_n,\ q_n^s \rightharpoonup q & \textit{weakly in} & L^2(0,T;H^1(D)), \\ q_n,\ q_n^s \longrightarrow q & \textit{strongly in} & L^2((0,T)\times D), \end{array} \tag{4.31}$$

$$Q_n \rightharpoonup hq \quad \textit{weakly in} \ \ H^1(0,T;H^{-1}(D)), \tag{4.32}$$

$$u_n \rightharpoonup u \quad \textit{weakly in} \ \ H^1((0,T)\times(0,L)) \tag{4.33}$$

as $n \longrightarrow \infty$.

Proof: The convergences (4.27), (4.28), (4.31), (4.32), (4.33) can be found in [9] or [26, page 47]. We only prove here strong convergences of $\boldsymbol{U}_n,\ \boldsymbol{U}_n^s,\ Q_n,\ Q_n^s$ in the corresponding spaces, cf. (4.29), (4.30), (4.31). To show the compactness of $\boldsymbol{U}_n$ in $L^p(0,T,L^p(D))$ we use the Lions-Aubin lemma.

Note that

$$W^{1,p}(D) \Subset L^p(D) \subset (W^{1,p}(D))^*,$$

where imbedding $W^{1,p}(D)$ into $L^p(D)$ is compact, imbedding $L^p(D)$ into $(W^{1,p}(D))^*$ is continuous and $W^{1,p}(D)$ and $(W^{1,p}(D))^*$ are reflexive spaces, see [1, Adams]. According to the Lions-Aubin lemma the imbedding of the space

$$\mathcal{W} = \{\boldsymbol{U}_n \in L^p(0,T;\boldsymbol{V}),\ \partial_t \boldsymbol{U}_n \in L^q(0,T;(W^{1,p}(D))^*)\}$$

into $L^p(0,T;L^p(D))$ is compact, where $\frac{1}{q}+\frac{1}{\alpha_1}=1,\ \alpha_1=\frac{p^2-p}{p^2-p-1}$ for $p\in \big((1+\sqrt{5})/2,2\big)$ and $\alpha_1=p$ if $p\ge 2$. This implies that

$$\boldsymbol{U}_n \longrightarrow h\boldsymbol{u} \text{ strongly in } L^p(0,T;L^p(D)). \tag{4.34}$$

Since $p>(1+\sqrt{5})/2$ we get also strong convergence in $L^1((0,T)\times D)$.

Further, we show that

$$\boldsymbol{U}_n \longrightarrow h\boldsymbol{u} \text{ strongly in } \ L^r((0,T)\times D) \text{ for } 1\le r<7/3.$$

First, we use the Nirenberg-Gagliago inequality [12]

$$\|\varphi\|_3 \le c\|\varphi\|_{1,p}^{\theta}\|\varphi\|_2^{1-\theta},\ \ \frac{p}{6(p-1)} \le \theta \le 1, \tag{4.35}$$

and integrate it in time. Then the Hölder inequality yields

$$\|\boldsymbol{U}_n - h\boldsymbol{u}\|_{L^3((0,T)\times D)} \le C\|\boldsymbol{U}_n - h\boldsymbol{u}\|_{L^\infty(0,T;L^2(D))}^{1-\frac{p}{6(p-1)}}\|\boldsymbol{U}_n - h\boldsymbol{u}\|_{L^p(0,T;W^{1,p}(D))}^{\frac{p}{6(p-1)}}, \tag{4.36}$$

which gives the boundedness in $L^3((0,T)\times D)$. Now, the interpolation argument for r such that $1 \le r < 7/3$

$$\|\boldsymbol{U}_n - h\boldsymbol{u}\|^r_{L^r((0,T)\times D)} \le \|\boldsymbol{U}_n - h\boldsymbol{u}\|^{\frac{1}{3}}_{L^1((0,T)\times D)}\|\boldsymbol{U}_n - h\boldsymbol{u}\|^{\frac{3r-1}{3}}_{L^3((0,T)\times D)} \quad (4.37)$$

leads to the desired result in (4.29).

It remains to show the strong convergence of piecewise constant sequence $\{\boldsymbol{U}^s_n\}$. Since $|\boldsymbol{U}_n - \boldsymbol{U}^s_n| \le |h^i u^i - h^{i-1}u^{i-1}|$ for $t \in ((i-1)\Delta t, i\Delta t)$, we have from the first a priori estimate (4.8)

$$\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|_{L^2((0,T)\times D)} = \sqrt{\Delta t}\left(\sum_{i=1}^n \int_D |h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}|^2\right)^{1/2} \le C\sqrt{\Delta t}.$$

Moreover, similarly as above, from the interpolation argument (4.37), from (4.36) and using imbedding $L^2 \subset L^1$ we have $\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|_{L^r((0,T)\times D)} \le C\sqrt{\Delta t}^{\frac{1}{3r}}$. Consequently

$$\boldsymbol{U}^s_n \longrightarrow h\boldsymbol{u} \text{ strongly in } L^r((0,T)\times D) \text{ for } 1 \le r < 7/3. \quad (4.38)$$

Now it remains to show (4.30) for $\boldsymbol{U}^s_n$ if $p \ge 7/3$. Indeed, we have

$$\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|^p_{L^p(0,T;L^p(D))} = \int_0^T \|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|^p_p \le c\int_0^T \|\nabla\boldsymbol{U}_n - \nabla\boldsymbol{U}^s_n\|^{p-1}_2\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|_2,$$

that follows from Lemma 3.1 with $\theta = \frac{p-1}{p}$. Further, using the Hölder inequality we get

$$\begin{aligned}&\int_0^T \|\nabla\boldsymbol{U}_n - \nabla\boldsymbol{U}^s_n\|^{p-1}_2\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|_2 \\ &\le c\||\nabla\boldsymbol{U}_n| + |\nabla\boldsymbol{U}^s_n|\|^{p-1}_{L^p(0,T;L^2(D))}\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|_{L^p(0,T;L^2(D))}.\end{aligned}$$

The first term on the right hand side is bounded, see (4.21). The second term can be written in the form

$$\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|_{L^p(0,T;L^2(D))} = \left(\int_0^t \|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|_2\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|^{p-1}_2\right)^{1/p}$$

and upper bounded by $\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|^{1/p}_{L^2((0,T)\times D)}\||\boldsymbol{U}_n| + |\boldsymbol{U}^s_n|\|^{(p-1)/p}_{L^\infty(0,T;L^2(D))}$. Since $\|\boldsymbol{U}_n - \boldsymbol{U}^s_n\|_{L^2((0,T)\times D)} \le c\sqrt{(\Delta t)}$ we obtain that $\boldsymbol{U}^s_n \to \boldsymbol{U}_n$ in $L^p((0,T)\times D)$ and thus

$$\boldsymbol{U}^s_n \to h\boldsymbol{u} \quad \text{strongly in } L^p((0,T)\times D) \text{ for } p \ge 7/3. \quad (4.39)$$

This, together with (4.38) proves (4.30).

Further we consider the boundary integrals in (4.29). With the assistance of Lemma 3.2 for $r = p$ and the Hölder inequality we get

$$\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^p_{L^p((0,T)\times S)} \le \int_0^T c\|\nabla(\boldsymbol{U}_n^s - h\boldsymbol{u})\|_p \|\boldsymbol{U}_n^s - h\boldsymbol{u}\|_p^{p-1} + c\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|_p^p$$
$$\le c\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|_{L^p(0,T;W^{1,p}(D))}\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^{p-1}_{L^p((0,T)\times D)} + c\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^p_{L^p((0,T)\times D)},$$

This tends to zero due to (4.39) and (4.38) and implies the desired result from (4.29).

Analogously, we prove the strong convergence of $q_n^s \longrightarrow q$ in $L^2(0,T)\times D)$, cf. (4.31). In this case, we obtain from the Lions-Aubin lemma using the imbeddings $W^{1,2}(D) \Subset L^2(D) \subset W^{-1,2}(D)$ the strong convergence of q_n in $L^2(0,T;L^2(D))$. Since $|Q_n - Q_s| \le |h^i q^i - h^{i-1}q^{i-1}|$ for $t \in ((i-1)\Delta t, i\Delta t)$ we have from (4.8) also

$$\|Q_n - Q_n^s\|_{L^2((0,T)\times D)} = \sqrt{\Delta t}\left(\sum_{i=1}^n \int_D |h^i q^i - h^{i-1}q^{i-1}|^2\right)^{1/2} \le C\Delta t.$$

Letting $n \to \infty$ we get the strong convergence of q_n^s in $L^2(0,T;L^2(D))$. ■

4.2 Limiting process

Now we are ready to let $n \to \infty$ and with the assistance of Lemma 4.2 prove the existence of unsteady weak solution to our problem defined in (2.6).

Consider first smooth test functions $\boldsymbol{\psi} \in C^1([0,T]\times\overline{D})$, $\phi \in C(0,T;H^1(D))$, $\xi \in C(0,T;H_0^1(0,L))$. Then construct piecewise constant and piecewise linear test functions $\boldsymbol{\psi}_n$, $\boldsymbol{\psi}_n^s$, ϕ_n, ϕ_s, ξ_n^s. It is easy to verify that

$$\begin{aligned}&\boldsymbol{\psi}_n \to \boldsymbol{\psi} \quad \text{in} \quad H^1((0,T)\times D), \quad \boldsymbol{\psi}_n^s \to \boldsymbol{\psi} \quad \text{in} \quad L^\infty(0,T;C^1(\overline{D})),\\ &\phi_n^s \to \phi \quad \text{in} \quad L^2(0,T;H^1(D)) \quad \text{and} \quad \xi_n^s \to \xi \quad L^\infty(0,T;H_0^1(0,L)),\end{aligned} \tag{4.40}$$

see e.g., [26].

We consider identity (4.1) with $r = n$, where we put $\boldsymbol{\omega} = \boldsymbol{\psi}^i = \boldsymbol{\psi}(y, i\Delta t) \in \boldsymbol{V}$, $v = \phi^i \in H^1(D), \vartheta = \xi^i \in H_0^1(D)$, and replace

$$\Delta t \sum_{i=1}^n \int_D \frac{\partial \boldsymbol{U}_n}{\partial t}\boldsymbol{\psi}^i\,dy \quad \text{by} \quad -\int_{\Delta t}^T \int_D \boldsymbol{U}_n^s(t-\Delta t)\frac{\partial \boldsymbol{\psi}_n}{\partial t}(t)\,dy\,dt.$$

This yields

$$
\begin{aligned}
&\int_{\Delta t}^{T}\int_D \boldsymbol{U}_n^s(t-\Delta t)\frac{\partial \boldsymbol{\psi}_n}{\partial t}dydt = \\
&\int_0^T\int_D \Bigg\{ -\frac{\partial h_n}{\partial t}\frac{\partial (y_2\boldsymbol{u}_n^s)}{\partial y_2}\boldsymbol{\psi}_n^s + ((u_n^s,\boldsymbol{\psi}_n^s)) - h_n^s q_n^s \operatorname{div}\boldsymbol{\psi}_n^s \\
&+\left(h_n u_{1n}^s\left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_1} - \frac{y_2}{h_n^s}\frac{\partial h_n^s}{\partial y_1}\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}\right) + u_{2n}^s\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}\right)\boldsymbol{\psi}_n^s + \frac{h_n^s}{2}\,\boldsymbol{u}_n^s\boldsymbol{\psi}_n^s \operatorname{div}\boldsymbol{u}_n^s\Bigg\}\, dydt \\
&+\int_0^T\left\{\varepsilon\left\langle\frac{\partial Q_n}{\partial t},\phi_n^s\right\rangle dt + \int_D -\varepsilon\frac{\partial h_n}{\partial t}\frac{\partial(y_2 q_n^s)}{\partial y_2}\phi_n^s + \varepsilon a_1(q_n^s,\phi_n^s) + \operatorname{div}\boldsymbol{u}_n^s\phi_n^s\, dy\right. \\
&+\int_0^1 h(L,t)\left(q_{out}^{n,s} - \frac{1}{2}|u_{1n}^{\ s}|^2\right)\boldsymbol{\psi}_{1n}^{\ s}\ (L,y_2)\,dy_2 \qquad (4.41)\\
&-\int_0^1 h(0,t)\left(q_{in}^{n,s} - \frac{1}{2}|u_{1n}^{\ s}|^2\right)\boldsymbol{\psi}_{1n}^{\ s}\ (0,y_2)\,dy_2 \\
&+\int_0^L\left(q_w^{n,s} - \frac{1}{2}u_{2n}^{\ s}\frac{\partial h_n}{\partial t}\right)\boldsymbol{\psi}_{2n}^{\ s}\ (y_1,1)\,dy_1 \\
&\left.+\int_0^L\left(\frac{\kappa}{\rho}\left(u_{2n}^{\ s} - \lambda u^i - (1-\lambda)\frac{\partial h_n}{\partial t}\right)\boldsymbol{\psi}_{2n}^s + \frac{\varepsilon}{2}\frac{\partial h_n}{\partial t}\, q_n^s\phi_n^s\right)(y_1,1)\,dy_1\right\} dt \\
&+\int_0^L\Bigg\{\frac{\partial u_n}{\partial t}\xi_n^s + c\,\frac{\partial u_n^s}{\partial y_1}\frac{\partial \xi_n^s}{\partial y_1} + a\left(\int_0^t\frac{\partial u_n^s}{\partial y_1}(y_1,\tau)d\tau\right)\frac{\partial \xi_n^s}{\partial y_1} - a\frac{\partial^2 R_0}{\partial y_1^2}\xi_n^s \\
&\quad +b\left(\int_0^t u_n^s(y_1,\tau)d\tau\right)\xi_n^s + \frac{\kappa}{\rho E}\left(\lambda u_n^s + (1-\lambda)\frac{\partial h_n}{\partial t} - u_{2n}^{\ s}\right)\xi_n^s\Bigg\}(y_1)\,dy_1.
\end{aligned}
$$

In the above identity we will concentrate on the to convergence of particular terms. Limiting process in other terms is analogous or simpler. Letting $n\to\infty$ we have from (4.29), (4.32) and (4.40)

$$
\int_{\Delta t}^{T}\int_D \boldsymbol{U}_n^s(t-\Delta t)\frac{\partial \boldsymbol{\psi}_n}{\partial t}(t)\,dy\,dt \longrightarrow \int_0^T\int_D h\boldsymbol{u}(t)\frac{\partial \boldsymbol{\psi}}{\partial t}(t)\,dy\,dt,
$$

$$
\int_T\left\langle\frac{\partial Q_n}{\partial t},\phi_n^s\right\rangle_{H^1} dt \longrightarrow \int_T\left\langle\frac{\partial hq}{\partial t},\phi\right\rangle_{H^1} dt.
$$

Now we prove convergence in the nonlinear term $b(\cdot,\cdot,\cdot)$ defined in (2.5). Let us estimate

$$
\begin{aligned}
&\left|\int_0^T b(\boldsymbol{u}_n^s,\boldsymbol{u}_n^s,\boldsymbol{\psi}_n^s) - b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi})dt\right| \le \\
&\underbrace{\int_0^T |b(\boldsymbol{u}_n^s,\boldsymbol{u}_n^s,\boldsymbol{\psi}_n^s) - b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi}_n^s)|dt}_{[1]} + \underbrace{\int_0^T |b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi}_n^s) - b(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi})|dt}_{[2]}
\end{aligned}
$$

According to Lemma 3.7 the term [1] is equal to $\frac{1}{2}\int_0^T B(\boldsymbol{u}_n^s, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s,) - B(\boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s, \boldsymbol{u}_n^s) - B(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_n^s) + B(\boldsymbol{u}, \boldsymbol{\psi}_n^s, \boldsymbol{u})dt$ and can be estimated as follows.

$$
\begin{aligned}
&2\int_0^T |b(\boldsymbol{u}_n^s, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_n^s)|dt \leq \\
&\int_0^T |B(\boldsymbol{u}_n^s - \boldsymbol{u}, \boldsymbol{\psi}_n^s, \boldsymbol{u}_n^s)| + |B(\boldsymbol{u}, \boldsymbol{\psi}_n^s, \boldsymbol{u}_n^s - \boldsymbol{u}_n)| \\
&\qquad + |B(\boldsymbol{u}_n^s - \boldsymbol{u}, \boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s)| + |B(\boldsymbol{u}, \boldsymbol{u}_n^s - \boldsymbol{u}, \boldsymbol{\psi}_n^s)|dt \leq \\
&\int_0^T \int_D \left| c(\boldsymbol{u}_n^s - \boldsymbol{u}) \left(\frac{\partial \boldsymbol{\psi}_n^s}{\partial y_1}\right)(\boldsymbol{u}_n^s + \boldsymbol{u}) \right| + c\left| \boldsymbol{\psi}_n^s \left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_1}\right)(\boldsymbol{u}_n^s - \boldsymbol{u}) \right| dydt \\
&+ \int_0^T \int_D \boldsymbol{\psi}_n^s \boldsymbol{u} \left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_1} - \frac{\partial \boldsymbol{u}}{\partial y_1}\right) + \ldots dydt \leq \\
&c\,\|\boldsymbol{\psi}_n^s\|_{L^\infty(0,T;C^1(\bar{D}))}\, \|\boldsymbol{u}_n^s - \boldsymbol{u}\|_{L^q((0,T)\times D)} \| |\boldsymbol{u}_n^s| + |\nabla \boldsymbol{u}_n^s| + |\boldsymbol{u}| \|_{L^p((0,T)\times D)} + \\
&\int_0^T \int_D \boldsymbol{\psi}_n^s \boldsymbol{u} \left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_1} - \frac{\partial \boldsymbol{u}}{\partial y_1}\right) + \ldots dydt
\end{aligned}
$$

From the first a priori estimate and weak convergences in $L^p(0,T;\boldsymbol{V})$, cf. (4.29) we have that term [1] goes to 0. It is easy to prove for $p \geq 2$. In what follows, we will consider $p \in ((1+\sqrt{5})/2, 2)$ and prove that [1] converges to 0. In this case $q = \frac{p}{p-1} \in (2, 2.618)$. Now, for the first term on the right hand side of above expression we use the Hölder inequality

$$
\|\varphi\|_{L^q}^q \leq c\|\varphi\|_{L^2}^a \|\varphi\|_{L^{\frac{2(q-a)}{2-a}}}^{q-a} \tag{4.42}
$$

for $0 < a < 2,\ a < q$. Than $\frac{2(q-a)}{2-a} \leq 3$ for $q < 3 - \frac{a}{2}$. Therefore setting, for example, $a = 1/10$ we obtain on the base of boundedness in $L^3((0,T)\times D)$, cf. (4.36), and due to the strong convergence in $L^2((0,T)\times D)$, cf. (4.29), that $\|\boldsymbol{u}_n^s - \boldsymbol{u}\|_{L^q((0,T)\times D)} \to 0$ for $q \in (2, 2.618)$. For the second term on the right hand side of above expression we use the fact that gradients converge weakly in $L^p((0,T)\times D)$. Since $L^3 \subset L^{\frac{p}{p-1}}$ for $p \geq 3/2$ and due to (4.36), we get weak convergence also for $p \in ((1+\sqrt{5})/2, 2)$.

The second term [2] can be estimated from above in the following way

$$
\begin{aligned}
&\int_0^T |b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_n^s) - b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi})|dt = c\int_0^T \int_D \left| \boldsymbol{u} \frac{\partial \boldsymbol{u}}{\partial y_1}(\boldsymbol{\psi}_n^s - \boldsymbol{\psi}) \right| dydt \leq \\
&c\|\boldsymbol{u}\|_{L^q((0,T)\times D)} \left\| \frac{\partial \boldsymbol{u}}{\partial y_1} \right\|_{L^p((0,T)\times D)} \|\boldsymbol{\psi}_n^s - \boldsymbol{\psi}\|_{L^\infty(0,T;C(D))}.
\end{aligned}
$$

Thus, from (4.40), from the fact that $L^3 \subset L^{\frac{p}{p-1}}$ for $p \geq 3/2$ and due to the boundedness in $L^3((0,T)\times D)$, cf. (4.36) we get also the convergence of term [2] to 0.

Now we consider the convergence in the viscous term

$$\int_0^T ((\boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s)) \to \int_0^T ((\boldsymbol{u}, \boldsymbol{\psi})) \quad \forall \boldsymbol{\psi} \in C^1([0,T] \times \overline{D}). \tag{4.43}$$

We prove only the convergence $\int_0^T ((\boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s)) \to \int_0^T ((\boldsymbol{u}, \boldsymbol{\psi}_n^s))$, the convergence result $\int_0^T ((\boldsymbol{u}, \boldsymbol{\psi}_n^s)) \to \int_0^T ((\boldsymbol{u}, \boldsymbol{\psi}))$ is straightforward and follows from (4.40). We know that $\boldsymbol{u}_n^s \in L^p(0,T;\boldsymbol{V})$, $\hat{e}(\boldsymbol{u}) \in L^p(0,T;L^p)$ and

$$\boldsymbol{u}_n^s \rightharpoonup \boldsymbol{u} \text{ in } L^p(0,T;\boldsymbol{V}), \qquad \hat{e}(\boldsymbol{u}_n^s) \rightharpoonup \hat{e}(\boldsymbol{u}) \text{ in } L^p(0,T;L^p(D)).$$

Let us define the operator $\mathcal{A}(\hat{e}(\boldsymbol{u})) \in L^q(0,T;(L^p(D))^*)$ now in the following way

$$\langle \mathcal{A}(\hat{e}(\boldsymbol{u})), \hat{e}(\boldsymbol{\psi}) \rangle = \int_0^T \int_D \tau_{ij}(\hat{e}(\boldsymbol{u})) \hat{e}_{ij}(\boldsymbol{\psi}) = \int_0^T ((\boldsymbol{u}, \boldsymbol{\psi})),$$

see also (2.4). As before, with use of [14, Lemma 1.19] we know that operator $\mathcal{A}$ is monotonous. Moreover, the operator $\mathcal{A}$ is bounded (4.18) and converges weakly $\mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)) \rightharpoonup f$ in $L^q(0,T;(L^p(D))^*)$. Analogously as above we have according to the Minty-Browder theorem that $f = \mathcal{A}(\hat{e}(\boldsymbol{u}))$ and

$$\mathcal{A}(\hat{e}(\boldsymbol{u}_n^s)) \rightharpoonup \mathcal{A}(\hat{e}(\boldsymbol{u})) \text{ in } L^q(0,T;(L^p(D))^*),$$

which implies (4.43). The limiting process in other terms is easier and it is a direct consequence of Lemma 4.2.

After preparation above, we let $n \to \infty$ in (4.41), where we firstly consider smooth test functions $\boldsymbol{\varpi} = \boldsymbol{\varpi}_\varepsilon = (\boldsymbol{\psi}_\varepsilon, \phi_\varepsilon, \xi_\varepsilon)$ in (4.41), see (4.40) and the limiting process above. Due to the approximation of the Sobolev functions by smooth functions we have that $\boldsymbol{\varpi}_\varepsilon \to \boldsymbol{\varpi}$ strongly in the respective spaces, i.e. $\boldsymbol{\psi}_\varepsilon \to \boldsymbol{\psi}$ strongly in $H^1(0,T;\boldsymbol{V})$, $\phi_\varepsilon \to \phi$ strongly in $L^2(0,T;H^1(D))$ and $\xi_\varepsilon \to \xi$ strongly in $L^2(0,T;H_0^1(0,L))$. Thus we limit in test functions, i.e. $\boldsymbol{\psi}_\varepsilon \to \boldsymbol{\psi}$, $\phi_\varepsilon \to \phi$, $\xi_\varepsilon \to \xi$ now. For the limiting process in the viscous term we have

$$\int_0^T \int_D ((\boldsymbol{u}, \boldsymbol{\psi}_\varepsilon - \boldsymbol{\psi})) \le C \|\boldsymbol{\psi}_\varepsilon - \boldsymbol{\psi}\|_{L^p(0,T;W^{1,p}(D))} \to 0, \tag{4.44}$$

see estimates (4.18). Further, for nonlinear term we have according to Lemma 3.7

$$\int_0^T b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_\varepsilon - \boldsymbol{\psi}) \le c \int_0^T |B(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}_\varepsilon - \boldsymbol{\psi})| dt + \int_0^T |B(\boldsymbol{u}, \boldsymbol{\psi}_\varepsilon - \boldsymbol{\psi}, \boldsymbol{u})| dt \tag{4.45}$$

We chose the first term on the right hand side of (4.45) and show its convergence for $(1+\sqrt{5})/2 < p < 2$. In the case $p \ge 2$ the estimates are easier and we address to [26]. Indeed, for $(1+\sqrt{5})/2 < p < 2$ we have

$$
\begin{aligned}
\int_0^T |B(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi}_\varepsilon-\boldsymbol{\psi})|dt &\le c\int_0^T\int_D |\boldsymbol{u}||\nabla\boldsymbol{u}||\boldsymbol{\psi}_\varepsilon-\boldsymbol{\psi}|dydt \\
&\le c\int_0^T \|\boldsymbol{u}\|_{\frac{2p}{3p-4}}\|\nabla\boldsymbol{u}\|_p\|\boldsymbol{\psi}_\varepsilon-\boldsymbol{\psi}\|_{\frac{2p}{2-p}}\,dt \qquad (4.46)\\
&\le c\max_{t\in(0,T)}\|\boldsymbol{\psi}_\varepsilon-\boldsymbol{\psi}\|_{1,p}\int_0^T \|\nabla\boldsymbol{u}\|_p^{\frac{2-p}{p-1}}\|\boldsymbol{u}\|_2^{1-\frac{2-p}{p-1}}\|\nabla\boldsymbol{u}\|_p\,dt,\\
&\le c\max_{t\in(0,T)}\|\boldsymbol{\psi}_\varepsilon-\boldsymbol{\psi}\|_{1,p}\|\boldsymbol{u}\|^{1-\frac{2-p}{p-1}}_{L^\infty(0,T;L^2)}\|\boldsymbol{u}\|^{\frac{2-p}{p-1}}_{L^{\frac{p(2-p)}{(p-1)^2}}(0,T;W^{1,p})}\|\boldsymbol{u}\|_{L^p(W^{1,p})},
\end{aligned}
$$

where the Hölder inequality in space, the Nirenberg-Gagliargo inequality (3.6) and the Hölder inequality in time were used. Since $p(2-p)/(p-1)^2 \le p$ for $p \ge (1+\sqrt{5})/2$, we have that $L^p \subset L^{\frac{p(2-p)}{(p-1)^2}}$ and therefore $\int_0^T |B(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi}_\varepsilon-\boldsymbol{\psi})|dt \to 0$ in (4.46).

We do not show explicitly the limiting process in other terms since it is analogous or even easier. Thus, the limiting process in (4.41) is now completed.

4.2.1 Weak time derivative

It remains to show that the limit of $\partial_t(h_n\boldsymbol{u}_n)$ is $\partial_t(h\boldsymbol{u})$. Note, that for fixed n using discrete per partes we have

$$
\int_0^T\int_D \frac{\partial \boldsymbol{U}_n}{\partial t}\boldsymbol{\psi}_n^s\,dy\,dt = -\int_0^T\int_D \boldsymbol{U}_n^s\frac{\partial\boldsymbol{\psi}_n}{\partial t}\,dy\,dt
$$

Thus, after letting $n\to\infty$ in the previous section we obtain

$$
-\int_0^T\int_D h\boldsymbol{u}\frac{\partial\boldsymbol{\psi}}{\partial t} = \int_0^T \left\langle \chi,\,\boldsymbol{\psi}\right\rangle_{W^{1,p}}, \qquad (4.47)
$$

for every test function $\boldsymbol{\varpi} = (\boldsymbol{\psi},\phi,\xi)$ such that $\boldsymbol{\psi}\in H^1(0,T;\boldsymbol{V})$, $\phi \in L^2(0,T;H^1(D))$ and $\xi \in L^2(0,T;H_0^1(0,L))$. Here χ is the weak limit of $\partial_t\boldsymbol{U}_n$, see (4.19)

$$
\frac{\partial h_n\boldsymbol{u}_n}{\partial t} \rightharpoonup \chi \quad \text{in} \quad L^q(0,T;(W^{1,p}(D))^*).
$$

Note, that $-\int_D h\boldsymbol{u}\frac{\partial\boldsymbol{\psi}}{\partial t} = \left(h\boldsymbol{u},\frac{\partial\boldsymbol{\psi}}{\partial t}\right)$, where $\left(\cdot,\cdot\right)$ represents the scalar product in $L^2(D)$. By the Riesz theorem it could be identified with a functional in $(L^2(D))^*$. Since $\frac{\partial\boldsymbol{\psi}}{\partial t}$ belongs also to $W^{1,p}(D)$, it is possible to extend functional from $(L^2(D))^*$ to $(W^{1,p}(D))^*$ and represent the duality between

$W^{1,p}(D)$ and $(W^{1,p}(D))^*$ with respect to the scalar product in $L^2(D)$, see e.g., [8, Feistauer]. Therefore we can write

$$-\int_0^T \int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t} = -\int_0^T \left\langle h\boldsymbol{u}, \frac{\partial \boldsymbol{\psi}}{\partial t}\right\rangle_{W^{1,p}}. \tag{4.48}$$

Choose $\boldsymbol{\psi} = \boldsymbol{w}(x)\xi(t)$ such that $\boldsymbol{w} \in W^{1,p}(D),\ \xi \in C_0^1(0,T)$. From (4.47) and (4.48) we have that

$$-\int_0^T \left\langle h\boldsymbol{u}, \boldsymbol{w}\right\rangle_{W^{1,p}} \xi'(t) = \int_0^T \left\langle \chi, \boldsymbol{w}\right\rangle_{W^{1,p}} \xi(t) \tag{4.49}$$

and consequently we get that χ is the weak distributive derivative

$$\chi = \frac{\partial (h\boldsymbol{u})}{\partial t} \quad \text{in} \quad L^q(0,T;(W^{1,p}(D))^*).$$

Moreover, from (4.47), (4.48) and the above equality we deduce that

$$-\int_0^T \int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t} = \int_0^T \left\langle \frac{\partial h\boldsymbol{u}}{\partial t}, \boldsymbol{\psi}\right\rangle_{W^{1,p}} \tag{4.50}$$

for every $\boldsymbol{\psi} \in H^1(0,T;\boldsymbol{V}) \cap L^p(0,T;\boldsymbol{V})$.

In what follows, we will need also the above property with $\boldsymbol{\psi} = \boldsymbol{u}$. Note that $\boldsymbol{u}$ has a lack of time derivative in classical and also weak sense. Therefore it is not trivial to write

$$\frac{1}{2}\int_D |\boldsymbol{u}|^2(t)h(t) + \frac{1}{2}\int_D\int_0^t |\boldsymbol{u}|^2\frac{\partial h}{\partial t} = \int_0^t \left\langle \frac{\partial (h\boldsymbol{u})}{\partial t}, \boldsymbol{u}\right\rangle_{W^{1,p}} \tag{4.51}$$

However, note, that [26, Lemma 4.4] proves this property for pairing between $W^{1,2}(D)$ and $(W^{1,2}(D))^*$. In order to write this property for pairing between $W^{1,p}(D)$ and $(W^{1,p}(D))^*$, $p > (1+\sqrt{5})/2$, we use the fact that

$$\mathcal{W} = \left\{\boldsymbol{u} \in L^p(0,T;W^{1,p}(D));\ \partial_t \boldsymbol{u} \in L^q(0,T;(W^{1,p}(D))^*)\right\} \subset C(0,T;L^2(D)),$$

q is given by (4.19), (4.17), see [14, Lemma 1.2.45]. Using this result the property (4.51) follows easily.

Let us summarize the existence result in the following theorem.

Theorem 4.1.
Let ε, κ are fixed and $p > (1+\sqrt{5})/2$. Assume (3.1)–(3.4), (1.12), (1.13) and that for boundary pressure we have $q_{in}, q_{out} \in L^\infty((0,T);L^2(0,1))$, $q_w \in L^\infty((0,T);L^2(0,L))$.
Then there exists an approximated weak solution of problem (1.1)–(1.11)

transformed to the fixed domain, in the sense of integral identity (2.6). Moreover,

$$\frac{\partial(h\boldsymbol{u})}{\partial t} \in L^q((0,T);W^{1,p}(D)^*), \quad \frac{\partial hq}{\partial t} \in L^2((0,T);H^{-1}(D)),$$

such that

$$\int_0^T \Big\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi} \Big\rangle dt = \int_0^T \int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t} dy dt,$$

where q is given by $\frac{1}{q}+\frac{1}{\alpha_1}=1$, and $\alpha_1 = \frac{p^2-p}{p^2-p-1}$ for $p \in \big(\frac{1+\sqrt{5}}{2}, 2\big)$ and $\alpha_1 = p$ for $p \geq 2$.

5 Uniqueness and continuous dependence on data

This section has two parts. In the first we prove the uniqueness of weak solution. Second part is devoted to the continuous dependence of weak solution on data. This result means, that if the given data functions are close each to other, then also the two different weak solutions are close. We will see that this in fact implies the contractiveness of the iterative process with respect to the domain deformation.

5.1 Uniqueness

In this section we show the uniqueness of weak solution for $p \geq 2$. Let us address to work [14], where the uniqueness of the shear dependent flows in a fixed two-dimensional domain is shown only for $p \geq 2$. The difficulty in the case $p < 2$ lies in the estimate of nonlinear convective term.We use the standard technique, see [8], [9], [26] based on Gronwall's lemma. At the beginning we assume two possible weak solution of our problem on the domain given by the same deformation but different pressures $q_{\partial D}$ at the boundary. We will show in this subsection, that if these pressures do not differ, we get only one weak solution. In the proof we particularly concentrate on nonlinear viscous and nonlinear convective term, since these are different to previous works [9], [26].

Let $(\boldsymbol{u}^1, q^1, u^1)$ and $(\boldsymbol{u}^2, q^2, u^2)$ be two weak solution of initial value problem (2.6). We subtract weak formulation for both solution, use test functions $\Big(\boldsymbol{\psi}, \phi, \frac{E}{\rho}\xi\Big)$, where $\boldsymbol{\psi} = \boldsymbol{u}^1 - \boldsymbol{u}^2,\ \phi = q^1 - q^2,\ \xi = u^1 - u^2$ and after some straightforward but tedious manipulations we get using (4.51)

$$
\begin{aligned}
&-\frac{1}{2}\int_D \left(h|\boldsymbol{\psi}|^2 + \varepsilon h|\phi|^2\right)(t)\, dy - \frac{E}{2\rho}\int_0^L |\xi|^2(t)\, dy_1 \\
&+\int_0^t \left\{\tau_{ij}(\hat{e}(\boldsymbol{\psi}))\hat{e}_{ij}(\boldsymbol{\psi}) + \varepsilon a(\phi,\phi) + \frac{Ec}{\rho}\|\xi\|^2_{H_0^1(0,L)}\right\} ds \\
&+\frac{Ea}{2\rho}\int_0^L \left(\int_0^t \frac{\partial \xi}{\partial y^1} ds\right)^2 dy_1 + \frac{Eb}{2\rho}\int_0^L \left(\int_0^t \xi\, ds\right)^2 dy_1 \\
&+\int_0^t b(\boldsymbol{u}^1,\boldsymbol{u}^1,\boldsymbol{\psi}) - b(\boldsymbol{u}^2,\boldsymbol{u}^2,\boldsymbol{\psi})\, ds \qquad (5.1)\\
&+\int_0^t \left\{\frac{1}{2}\int_D \frac{\partial h}{\partial t}\left(|\boldsymbol{\psi}|^2 + \varepsilon|\phi|^2\right) dy + \frac{\kappa}{\rho}\int_0^L (\xi - \psi_2)(\lambda\xi - \psi_2)\, dy_1\right\} ds \\
&+\int_0^t \left\{\int_0^1 h(L)(q^1_{out} - q^2_{out})\psi_1(L) - h(0)(q^1_{in} - q^2_{in})\psi_1(0) dy_2\right\} ds \\
&+\int_0^t \int_0^L (q^1_w - q^2_w)\psi_2(1) - a\frac{\partial^2 R_0}{\partial y_1^2}\xi\, dy_1\, ds \;=\; 0,
\end{aligned}
$$

where we recall the notation from (2.5) and

$$
a_1(\phi,\phi) = \int_D h\left(\frac{\partial \phi}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_2}\frac{\partial \phi}{\partial y_2}\right)^2 + \frac{1}{h}\left(\frac{\partial \phi}{\partial y_2}\right)^2 dy.
$$

To estimate the right-hand side of (5.1), we first rewrite $b(\boldsymbol{w}^1,\boldsymbol{w}^1,\boldsymbol{\zeta}) - b(\boldsymbol{w}^2,\boldsymbol{w}^2,\boldsymbol{\zeta})$. According to Lemma 3.7 it is not difficult to verify analogously as before that

$$
\begin{aligned}
b(\boldsymbol{w}^1,\boldsymbol{w}^1,\boldsymbol{\zeta}) - b(\boldsymbol{w}^2,\boldsymbol{w}^2,\boldsymbol{\zeta}) = & \qquad (5.2)\\
&\frac{1}{2}B(\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{u}^1,\boldsymbol{\psi}) - \frac{1}{2}B(\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{\psi},\boldsymbol{u}^1)\\
&+\frac{1}{2}B(\boldsymbol{u}^2,\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{\psi}) - \frac{1}{2}B(\boldsymbol{u}^2,\boldsymbol{\psi},\boldsymbol{u}^1-\boldsymbol{u}^2)\\
&= \frac{1}{2}B(\boldsymbol{\psi},\boldsymbol{u}^1,\boldsymbol{\psi}) - \frac{1}{2}B(\boldsymbol{\psi},\boldsymbol{\psi},\boldsymbol{u}^1).
\end{aligned}
$$

Moreover, after applying coercitivity of forms $a_1(\cdot,\cdot)$, $((\cdot,\cdot))$, see Lemma 3.3, and 3.4 we obtain for $p \geq 2$ the following inequality:

$$\begin{aligned}
&\frac{\alpha}{2}\int_D \left(|\boldsymbol{\psi}|^2 + \varepsilon|\phi|^2\right)(t)\,dy + \frac{E}{2\rho}\int_0^L |\xi|^2(t)\,dy_1 \\
&+ \int_0^t \left\{ \delta\int_D |\nabla(\boldsymbol{\psi})|^2 + |\nabla(\boldsymbol{\psi})|^p + \frac{\alpha}{2+K^2}\varepsilon\int_D |\nabla\phi|^2 + \frac{Ec}{\rho}\|\xi\|^2_{H_0^1(0,L)} \right\} ds \\
&+ \frac{Ea}{2\rho}\int_0^L \left(\int_0^t \frac{\partial \xi}{\partial y^1} ds\right)^2 dy_1 + \frac{Eb}{2\rho}\int_0^L \left(\int_0^t \xi\,ds\right)^2 dy_1 \\
&\leq \int_0^t \frac{1}{2}|B(\boldsymbol{\psi},\boldsymbol{u}^1,\boldsymbol{\psi})| + \frac{1}{2}|B(\boldsymbol{\psi},\boldsymbol{\psi},\boldsymbol{u}^1)|\,ds \qquad (5.3)\\
&+ \int_0^t \left\{ \frac{1}{2}\int_D \left|\frac{\partial h}{\partial t}\right| \left(|\boldsymbol{\psi}|^2 + \varepsilon|\phi|^2\right) dy + \frac{\kappa}{\rho}\int_0^L \left(|\xi|^2 + |\psi_2|^2 + 2|\xi||\psi_2|\right) dy_1 \right\} ds \\
&+ \int_0^t \alpha^{-1} \left\{ \int_0^1 |q^1_{out} - q^2_{out}||\psi_1(L)| + |q^1_{in} - q^2_{in}||\psi_1(0)|dy_2 \right\} ds \\
&+ \int_0^t \int_0^L |q^1_w - q^2_w||\psi_2(1)| + a\left|\frac{\partial^2 R_0}{\partial y_1^2}\right| |\xi|\,dy_1\,ds.
\end{aligned}$$

Note, that from (5.1) we have for $\lambda = 1$ independence of constants appearing in the above estimate on κ, since then the positive term with κ could be omitted. Now, we estimate the right hand side of (5.3), where we point out only the term $\frac{1}{2}|B(\boldsymbol{\psi},\boldsymbol{u}^1,\boldsymbol{\psi})| + \frac{1}{2}|B(\boldsymbol{\psi},\boldsymbol{\psi},\boldsymbol{u}^1)|$. Other terms are simpler, we used there similar technique, in boundary terms the trace theorem has been used.

$$\begin{aligned}
&|B(\boldsymbol{\psi},\boldsymbol{u}^1,\boldsymbol{\psi})| + |B(\boldsymbol{\psi},\boldsymbol{\psi},\boldsymbol{u}^1)| = \qquad (5.4)\\
&\int_D (\boldsymbol{u}^1 - \boldsymbol{u}^2)\frac{\partial \boldsymbol{u}^1}{\partial y_1}(\boldsymbol{u}^1 - \boldsymbol{u}^2) + \ldots + (\boldsymbol{u}^1 - \boldsymbol{u}^2)\frac{\partial(\boldsymbol{u}^1 - \boldsymbol{u}^2)}{\partial y_1}\boldsymbol{u}_1 + \ldots dy \\
&\leq \|\nabla \boldsymbol{u}^1\|_2 \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_4^2 + \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,2}\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_4 \|\boldsymbol{u}^1\|_4
\end{aligned}$$

Now, according to Lemma 3.1, cf. (3.6), and due to the compact imbedding (3.22) we get from (5.4)

$$\begin{aligned}
&|B(\boldsymbol{\psi},\boldsymbol{u}^1,\boldsymbol{\psi})| + |B(\boldsymbol{\psi},\boldsymbol{\psi},\boldsymbol{u}^1)| \leq \\
&c\|\boldsymbol{u}^1\|_{1,2}\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,2}\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_2 + c\|\boldsymbol{u}^1\|_2^{\frac{1}{2}}\|\boldsymbol{u}^1\|_{1,2}^{\frac{1}{2}}\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,2}^{\frac{3}{2}}\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_2^{\frac{1}{2}}.
\end{aligned}$$

Using Young's inequalities for $p, q = 2$ and $p = 4/3,\ q = 4$ we get

$$\begin{aligned}
&|B(\boldsymbol{\psi},\boldsymbol{u}^1,\boldsymbol{\psi})| + |B(\boldsymbol{\psi},\boldsymbol{\psi},\boldsymbol{u}^1)| \leq \\
&\epsilon(1 + \|\boldsymbol{u}^1\|_2^{\frac{2}{3}})\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|^2_{1,2}, + C_\epsilon\|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_2^2\|\boldsymbol{u}^1\|^2_{1,2}
\end{aligned}$$

where ϵ is small enough.

Now we subtract the terms with ϵ coming from the estimation of non-linear viscous term from the left hand side of (5.3). We arrive at

$$g(t)+\int_0^t (\delta-\mu)\,\|\boldsymbol{\psi}\|_{1,2}^2+\delta\|\boldsymbol{\psi}\|_{1,p}^p ds+\left(\frac{\alpha}{2+K^2}\right)\varepsilon\int_0^t\|\phi\|_{1,2}^2 ds$$
$$+\frac{Ea}{2\rho}\int_0^L\left(\int_0^t\frac{\partial\xi}{\partial y^1}ds\right)^2+\frac{Eb}{2\rho}\left(\int_0^t\xi\;ds\right)^2 dy_1+\frac{Ec}{\rho}\int_0^t\|\nabla\xi\|_{L^2(0,L)}^2 ds$$
$$\le\;\vartheta(t)+\int_0^t r(s)g(s)\;ds, \tag{5.5}$$

where $\mu=\epsilon(1+\|\boldsymbol{u}^1\|^{2/3}_{L^\infty(0,T:L^2(D))})$,

$$g(t)=\frac{\alpha}{2}\int_D\left(|\boldsymbol{\psi}|^2+\varepsilon|\phi|^2\right)(t)\;dy+\frac{E}{2\rho}\int_0^L|\xi|^2(t)\;dy_1, \tag{5.6}$$

$$\vartheta(t)\equiv C_1\int_0^t\left(\|q^1_{out}-q^2_{out}\|^2_{L^2(S_{out})}+\|q^1_{in}-q^2_{in}\|^2_{L^2(S_{in})}+\|q^1_w-q^2_w\|^2_{L^2(S_w)}\right)(s)ds,$$

$C_1=C_1(\alpha)$ and integrable function

$$r(s)\equiv C_\epsilon(E,\rho,\kappa,\alpha)\Big(\|\nabla\boldsymbol{u}^1\|_2^2+1+\|\partial_t h\|_\infty+a\left\|\frac{\partial R_0}{\partial y_1}\right\|_{L^2(0,L)}\Big)(s),$$

Note that after omitting positive terms in (5.5) we have $g(t)\le\vartheta(t)+\int_0^t r(s)g(s)ds$. Gronwall's lemma [8, Lemma 8.2.29] yields

$$g(t)\le\vartheta(t)+\int_0^t\vartheta(s)r(s)e^{\int_s^t r(\tau)d\tau}ds\le\vartheta(t)C_2,$$

where $C_2=1+\int_0^t r(s)ds\exp(\int_0^t r(\tau)d\tau)$. Using the above inequality and estimating $g(s)$ on the right hand side of (5.5) we get

$$\vartheta(t)+\int_0^t r(s)g(s)\;ds\le C_3\vartheta(t),$$

where $C_3=1+C_2\int_0^t r(s)ds$ depends on $\int_0^t r(s)ds$.

We can summarize the result in the following theorem.

Theorem 5.1 (Uniqueness).
Let $p\ge 2$ and $(\boldsymbol{u}^1,q^1,u^1)$, $(\boldsymbol{u}^2,q^2,u^2)$ be two weak solutions of the initial boundary value problem (1.1)–(1.11) transformed to the fixed domain, in the sense of Definition 2.1. Let the corresponding data functions be h, q^1_{in}, q^1_w,

q^1_{out} and q^2_{in}, q^2_w, q^2_{out}, respectively. Suppose that the assumptions (1.12), (1.13) and (3.1)–(3.4) hold. Then for almost all $t \in [0,T]$ it holds:

$$\begin{aligned}
&\frac{\alpha}{2}\left(\|\boldsymbol{u}^1-\boldsymbol{u}^2\|_2^2+\varepsilon\|q^1-q^2\|_2^2\right)(t) + \frac{E}{2\rho}\|u^1-u^2\|^2_{L^2(0,L)}(t) \qquad (5.7)\\
&+\int_0^t(\delta-\mu)\|\boldsymbol{u}^1-\boldsymbol{u}^2\|^2_{1,2}+\delta\|\boldsymbol{u}^1-\boldsymbol{u}^2\|^p_{1,p}ds+\\
&\left(\frac{\alpha}{2+K^2}\right)\varepsilon\int_0^t\|q^1-q^2\|^2_{1,2}+\frac{Ec}{\rho}\|u^1-u^2\|^2_{H^1(0,L)}ds\\
&+\frac{Ea}{2\rho}\int_0^L\left(\int_0^t\frac{\partial(u^1-u^2)}{\partial y^1}ds\right)^2+\frac{Eb}{2\rho}\left(\int_0^t(u^1-u^2)\,ds\right)^2dy_1\\
&\le C\int_0^t\left(\|q^1_{out}-q^2_{out}\|^2_{L^2(S_{out})}+\|q^1_{in}-q^2_{in}\|^2_{L^2(S_{in})}+\|q^1_w-q^2_w\|^2_{L^2(S_w)}\right)ds,
\end{aligned}$$

where $C=C_\epsilon(E,\rho,\kappa,\alpha)(1+k+e^kk^2)$, $k=\int_0^t r(s)ds$, $r(t)=\|\nabla\boldsymbol{u}^1\|_2^2(t)+1+\|\partial_t h\|_\infty(t)+a\left\|\frac{\partial R_0}{\partial y_1}\right\|_{L^2(0,L)}(t)$. Consequently for $q^1_{\partial D}=q^2_{\partial D}$ we get the uniqueness of weak solution (2.6) to our problem.

5.2 Continuous dependence on data

In this section we show the continuous dependence of weak solution on data; i.e. the given domain deformation and the pressures on the boundary $q_{\partial D}$. We will explain the proof in more details only for the nonlinear viscous term, since this is a new part of our model. Other terms can be handled as in [9], [26]. Throughout this section we assume that $p \ge 2$. Firstly we introduce the following helpful assertions.

Lemma 5.1.
Let properties (1.12) and (1.13) be satisfied, then

$$\int_D|\hat{e}_h(\boldsymbol{u})|+\int_D\left|\frac{\partial\hat{e}_h(\boldsymbol{u})}{\partial h}\right|\le C(\alpha,K,p)\int_D|\nabla\boldsymbol{u}| \qquad (5.8)$$

Proof: The assertions follow from the definition of $\hat{e}$, (2.4) and properties of function h. ■

Let us consider two solutions $(\boldsymbol{u}^1,q^1,u^1)$, $(\boldsymbol{u}^2,q^2,u^2)$ obtained for boundary data $q^1_{\partial D}$, $q^2_{\partial D}$ on $\Omega(h^1),\Omega(h^2)$, respectively. In [26, Chapter 3.7] the proof of continuous dependence on data is proven for bilinear viscous form. In this section we concentrate on differences in this proof for our nonlinear viscous form (2.4). Since $\hat{e}$ is dependent on the domain deformation function h, and $h^1 \ne h^2$, we will explicitly point out this dependence $\hat{e}(\boldsymbol{u})=\hat{e}(h,\boldsymbol{u})=\hat{e}_h(\boldsymbol{u})$.

Let us denote $\boldsymbol{\psi}=\boldsymbol{u}^1-\boldsymbol{u}^2$ and subtract the weak formulation for both solutions using test functions $\boldsymbol{\psi}, q^1-q^2, \frac{E}{\rho}(u^1-u^1)$. In what follows we will concentrate on the following two terms

$$\int_0^t \int_D \left\{ \tau_{ij}(\hat{e}_{h^1}(\boldsymbol{u}^1))\hat{e}_{h^1 ij}(\boldsymbol{\psi}) - \tau_{ij}(\hat{e}_{h^2}(\boldsymbol{u}^2))\hat{e}_{h^2 ij}(\boldsymbol{\psi}) \right\} dy\, ds, \tag{5.9}$$

$$\int_0^t \left\langle \frac{\partial(h^1\boldsymbol{u}^1)}{\partial t}, \boldsymbol{\psi} \right\rangle - \left\langle \frac{\partial(h^2\boldsymbol{u}^2)}{\partial t}, \boldsymbol{\psi} \right\rangle ds = \int_0^t \left\langle \frac{\partial(h^1\boldsymbol{\psi})}{\partial t} + \frac{\partial \boldsymbol{u}^2}{\partial t}(h^1 - h^1), \boldsymbol{\psi} \right\rangle ds. \tag{5.10}$$

All other terms have already been studied in [26]. According to the property (4.51), the first term on the right hand side of (5.10) equals to

$$\int_0^t \left\langle \frac{\partial}{\partial t}(h^1\boldsymbol{\psi}), \boldsymbol{\psi} \right\rangle = \frac{1}{2}\int_D |\boldsymbol{\psi}|^2(t)h_1(t) + \frac{1}{2}\int_D \int_0^t |\boldsymbol{\psi}|^2 \frac{\partial h_1}{\partial t} dyds. \tag{5.11}$$

Due to the properties of h, (1.12) and (1.13), we can estimate the right hand side terms $1/2\int_D |\boldsymbol{\psi}|^2(t)h_1(t) \geq \alpha/2\|\boldsymbol{\psi}\|_2^2(t)$, $1/2\int_D \int_0^t |\boldsymbol{\psi}|^2 \frac{\partial h_1}{\partial t} dyds \geq K/2\int_0^t \|\boldsymbol{\psi}(s)\|_2^2 ds$.

The second term on the right hand can be estimated as follows

$$\begin{aligned} &\int_0^t \left\langle \boldsymbol{u}^2 \frac{\partial}{\partial t}(h^1 - h^2), \boldsymbol{\psi} \right\rangle + \left\langle (h^1 - h^2)\frac{\partial \boldsymbol{u}^2}{\partial t} \right\rangle ds \\ &\leq \|\partial_t h^1 - \partial_t h^2\|_{W^{1,\infty}((0,T)\times(0,L))} \|\boldsymbol{u}^2\|_{L^2(Q_T)} \|\boldsymbol{\psi}\|_{L^2(Q_t)} ds + \qquad (5.12) \\ &\quad \|h^1 - h^2\|_{W^{1,\infty}((0,T)\times(0,L))} \left\| \frac{\partial \boldsymbol{u}^2}{\partial t} \right\|_{L^q(0,T;(W^{1,p})^*)} \|\boldsymbol{\psi}\|_{L^p(0,T;W^{1,p})} \\ &\leq \epsilon \int_0^t \|\boldsymbol{\psi}\|_{1,p}^p ds + \frac{1}{2}\int_0^t \|\boldsymbol{\psi}\|_2^2 ds + \frac{1}{2}\|h^1 - h^2\|^2_{W^{1,\infty}((0,T)\times(0,L))} \|\boldsymbol{u}^2\|^2_{L^2(Q_T)} \\ &\quad + C_\epsilon \|h^1 - h^2\|^q_{W^{1,\infty}((0,T)\times(0,L))} \left\| \frac{\partial \boldsymbol{u}^2}{\partial t} \right\|^q_{L^q(0,T;(W^{1,p})^*)}, \end{aligned}$$

$q = \frac{p}{p-1}$, $Q_T = (0,T) \times D$, $Q_t = (0,t) \times D$.

In order to estimate (5.9), we use the mean value theorem to express $\tau_{ij}(\hat{e}_{h^2}(\boldsymbol{u}^2))$ dependent on h^2 by means of h^1:

$$\begin{aligned} &\tau_{ij}(\hat{e}_{h^2}(\boldsymbol{u}^2)) = \\ &\tau_{ij}(\hat{e}_{h^1}(\boldsymbol{u}^2)) + \frac{\partial \tau_{ij}}{\partial \hat{e}} \left(\theta \hat{e}_{h^1}(\boldsymbol{u}^2) + (1-\theta)\hat{e}_{h^2}(\boldsymbol{u}^2) \right) [\hat{e}_{h^2 ij}(\boldsymbol{u}^2) - \hat{e}_{h^1 ij}(\boldsymbol{u}^2)], \end{aligned}$$

$0 \leq \theta \leq 1$. Analogously we can express $\hat{e}_{h^2}(\boldsymbol{u}^2)$ as

$$\hat{e}_{h^2}(\boldsymbol{u}^2) = \hat{e}_{h^1}(\boldsymbol{u}^2) + \partial_h \hat{e}_{(\theta h^1 + (1-\theta)h^2)}(\boldsymbol{u}^2)[h^2 - h^1],$$

and therefore (5.9) is equal to

$$
\begin{aligned}
&\int_0^t \int_D \left\{\tau_{ij}(\hat e_{h^1}(\boldsymbol u^1)) - \tau_{ij}(\hat e_{h^2}(\boldsymbol u^2))\right\}\left[\hat e_{h^1 ij}(\boldsymbol\psi)\right] \\
&\qquad - \tau_{ij}(\hat e_{h^2}(\boldsymbol u^2))\left[\frac{\partial \hat e_{\bar\theta_h ij}}{\partial h}(\boldsymbol\psi)[h^2-h^1]\right] dy\,ds = \\
&\int_0^t \int_D \left(\tau_{ij}(\hat e_{h^1}(\boldsymbol u^1)) - \tau_{ij}(\hat e_{h^1}(\boldsymbol u^2)) - \frac{\partial \tau_{ij}}{\partial \hat e}(\bar\theta_{\hat e_h})[\hat e_{h^2 ij}(\boldsymbol u^2) - \hat e_{h^1 ij}(\boldsymbol u^2)]\right) \times \\
&\qquad \left[\hat e_{h^1 ij}(\boldsymbol\psi)\right] - \tau_{ij}(\hat e_{h^2}(\boldsymbol u^2))\left[\frac{\partial \hat e_{\bar\theta_h ij}}{\partial h}(\boldsymbol\psi)[h^2-h^1]\right] dy\,ds = \\
&\int_0^t \int_D \underbrace{\left\{\tau_{ij}(\hat e_{h^1}(\boldsymbol u^1)) - \tau_{ij}(\hat e_{h^1}(\boldsymbol u^2))\right\}\left[\hat e_{h^1 ij}(\boldsymbol\psi)\right]}_{[1]} \\
&\qquad - \underbrace{\frac{\partial \tau_{ij}}{\partial \hat e}(\bar\theta_{\hat e_h})\left[\frac{\partial \hat e_{\bar\theta_h ij}}{\partial h}(\boldsymbol u^2)[h^2-h^1]\right]\left[\hat e_{h^1 ij}(\boldsymbol\psi)\right]}_{[2]} \\
&\qquad - \underbrace{\tau_{ij}(\hat e_{h^2}(\boldsymbol u^2))\left[\frac{\partial \hat e_{\bar\theta_h ij}}{\partial h}(\boldsymbol\psi)[h^2-h^1]\right]}_{[3]} dy\,ds,
\end{aligned}
$$

where $\bar\theta_{\hat e_h} = \theta \hat e_{h^1}(\boldsymbol u^2) + (1-\theta)\hat e_{h^2}(\boldsymbol u^2)$ and $\bar\theta_h = \theta h^1 + (1-\theta)h^2$. Now, we estimate the terms [1], [2], [3].

Term [2] can be estimated from above as follows

$$
\begin{aligned}
[2] \;\le\;& \|h^1-h^2\|_{L^\infty((0,T)\times(0,L))} \int_0^t \int_D \left|\frac{\partial \tau}{\partial \hat e}(\bar\theta_{\hat e_h})\right| \left|\frac{\partial \hat e_{\bar\theta_h}}{\partial h}(\boldsymbol u^2)\right| |\hat e_{h^1}(\boldsymbol\psi)|\; dy\; ds \\
& \text{further using (3.4), (5.8), Hölder's and Young's inequality} \\
\le\;& C(K,\alpha)\|h^1-h^2\| \int_0^t \int_D C_2(1+|\bar\theta_{\hat e_h}|)^{p-2}|1+\nabla \boldsymbol u^2||\nabla\boldsymbol\psi|\; dy\; ds \\
\le\;& C_3(K,\alpha)\|h^1-h^2\| \int_0^t \left(\int_D |1+|\nabla \boldsymbol u^2||^{p-1\cdot\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_D |\nabla\boldsymbol\psi|^p\right)^{\frac{1}{p}}\; ds \\
=\;& C_3(K,\alpha)\|h^1-h^2\| \int_0^t \|1+|\nabla \boldsymbol u^2|\|_p^{p-1}\|\nabla\boldsymbol\psi\|_p\; ds \\
\le\;& \epsilon\int_0^t \|\nabla\boldsymbol\psi\|_p^p ds + C_\epsilon \|h^1-h^2\|_{L^\infty((0,T)\times(0,L))}^{\frac{p}{p-1}} \int_0^t \|1+|\nabla \boldsymbol u^2|\|_p^p ds,
\end{aligned}
$$

where $\|h^1-h^2\| = \|h^1-h^2\|_{L^\infty((0,T)\times(0,L))}$.
From the property

$$
|\tau_{ij}(\hat e(\boldsymbol u))| \le C_5(1+|\hat e(\boldsymbol u)|)^{p-1}, \qquad (5.13)
$$

which is derived from (3.1), (3.4), cf. [14, Lemma 1.19], or the proof of

Lemma 3.5, the estimate of the term [3] can be derived

$$[3] \leq C_5(K,\alpha)\|h^1 - h^2\|_{L^\infty((0,T)\times(0,L))} \int_0^t \int_D (1 + |\hat{e}(\boldsymbol{u}^2)|)^{p-1} |\nabla \boldsymbol{\psi}| \; dy \; ds.$$

Similarly as above using the Hölder and the Young inequalities we get

$$[3] \leq \epsilon \int_0^t \|\nabla \boldsymbol{\psi}\|_p^p ds + C_{\epsilon_1} \|h^1 - h^2\|_{L^\infty((0,T)\times(0,L))}^{\frac{p}{p-1}} \int_0^t 1 + \|\nabla \boldsymbol{u}^2\|_p^p ds.$$

We can conclude that both terms [2] and [3] can be estimated by

$$\begin{aligned}[2] + [3] \leq \epsilon \int_0^t \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,p}^p ds \\ + C_\epsilon \|h^1 - h^2\|_{L^\infty((0,T)\times(0,L))}^{\frac{p}{p-1}} \int_0^t 1 + \|\boldsymbol{u}^2|\|_{1,p}^p ds,\end{aligned} \tag{5.14}$$

where we have replaced $\boldsymbol{\psi} = h^1(\boldsymbol{u}^1 - \boldsymbol{u}^2) + \boldsymbol{u}^2(h^1 - h^2)$ and $C_{\epsilon_1} = C(\frac{1}{4\epsilon_1}, K, \alpha)$.

We estimate now the term [1]. We omit the subscript h^1 in [1] for the sake of simplicity, since now both $\hat{e}, \tau$ are considered with respect to h^1. According to Lemma 3.4 item 2. we have

$$\begin{aligned}[1] \;\; &= \;\; \int_0^t \int_D \{\tau_{ij}(\hat{e}(\boldsymbol{u}^1)) - \tau_{ij}(\hat{e}(\boldsymbol{u}^2))\} \hat{e}(\boldsymbol{u}^1 - \boldsymbol{u}^2) \\ &\geq c_\delta \int_0^t \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,2}^2 + \|\boldsymbol{u}^1 - \boldsymbol{u}^2\|_{1,p}^p \; ds,\end{aligned} \tag{5.15}$$

where $c_\delta = \delta \alpha c$, c coming from the generalized Korn inequality, δ is coercivity constant from Lemma 3.4.

Now, taking into account the boundedness of terms [1], [2], [3] and (5.11), (5.12), after subtracting all "ϵ-terms" in (5.14), (5.12) from the right hand side of (5.15) we get

$$\begin{aligned}&\frac{\alpha}{2}\|\boldsymbol{\psi}\|_2^2(t) + (c_\delta - \epsilon) \int_0^t \|\boldsymbol{\psi}\|_{1,2}^2 + \|\boldsymbol{\psi}\|_{1,p}^p ds \\ &\leq \frac{C}{2} \int_0^t \|\boldsymbol{\psi}\|_2^2 ds + \omega(t) \sum_{j=2,\frac{p}{p-1}} \|h^1 - h^2\|_{W^{1,\infty}((0,T)\times(0,L))}^j,\end{aligned} \tag{5.16}$$

where

$$\omega(t) = C_\epsilon \|\partial_t \boldsymbol{u}^2\|_{L^q(0,T;(W^{1,p})^*)}^q + C_\epsilon \int_0^t 1 + \|\boldsymbol{u}^2\|_{1,p}^p + \|\boldsymbol{u}^2\|_2^2 ds.$$

After ommiting positive terms on the left hand side of (5.16) we obtain

$$g(t) \leq \vartheta(t) + C \int_0^t g(s), \tag{5.17}$$

where $g(t) = \frac{1}{2}\|\boldsymbol{\psi}\|_2^2(t)$ and

$$\vartheta(t) \;=\; \omega(t) \sum_{j=2,\frac{p}{p-1}} \|h^1 - h^2\|^j_{W^{1,\infty}((0,T)\times(0,L))},\ \omega(t) \to 0 \text{ for } t \to 0.$$

Here we can stop our consideration, because analogical method like in previous section is used and the Gronwall lemma is applied to (5.17). We should note also that estimates of nonlinear convective terms

$$b_{h1}(\boldsymbol{u}^1, \boldsymbol{u}^1, \boldsymbol{\psi}) - b_{h^2}(\boldsymbol{u}^2, \boldsymbol{u}^2, \boldsymbol{\psi})$$

must be involved.This estimation are however done in the previous section, see (5.2) and (5.4). Note that some additional error terms $|h^1 - h^2|$ coming from the fact that $h^1 \neq h^2$ must be involved. Since this is done in [26], [9], we will not repeat it here.

Let us summarize the main result of this section.

Theorem 5.2 (Continuous dependence on data)**.**
Assume that we have two weak solutions $(\boldsymbol{u}^1, q^1, u^1)$, $(\boldsymbol{u}^2, q^2, u^2)$ *of the initial boundary value problem (1.1)–(1.11) in the sense of Definition 2.1. Let* $p \geq 2$ *and the corresponding data be* h^1, h^2, q^1_{in}, q^1_w, q^1_{out} *and* q^2_{in}, q^2_w, q^2_{out}, *respectively. Moreover suppose that the assumptions (1.12), (1.13) and (3.1)–(3.4) hold. Then for almost all* $t \in [0,T]$ *it holds*

$$\begin{aligned}
&\frac{1}{2}\left(\|h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2\|_2^2 + \varepsilon\|h^1 q^1 - h^2 q^2\|_2^2\right)(t)\, dy + \frac{E}{2\rho}\|u^1 - u^2\|^2_{L^2(0,L)}(t)\, dy_1 \\
&+\bar{c_\delta}\int_0^t \|h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2\|^p_{1,p} + \|h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2\|^2_{1,2} ds \\
&+\left(\frac{\alpha^2}{2+K^2}\right)\varepsilon\int_0^t \|h^1 q^1 - h^2 q^2\|^2_{1,2} + \frac{Ec}{\rho}\|u^1 - u^2\|^2_{H^1(0,L)} ds \qquad (5.18)\\
&+\frac{Ea}{2\rho}\int_0^L \left(\int_0^t \frac{\partial(u^1-u^2)}{\partial y^1} ds\right)^2 + \frac{Eb}{2\rho}\left(\int_0^t (u^1 - u^2)\, ds\right)^2 dy_1 \\
&\leq\ C_1 \int_0^t \left(\|q^1_{out} - q^2_{out}\|^2_{L^2(S_{out})} + \|q^1_{in} - q^2_{in}\|^2_{L^2(S_{in})} + \|q^1_w - q^2_w\|^2_{L^2(S_w)}\right) ds \\
&+\omega(t)\|h^1 - h^2\|^2_{W^{1,\infty}((0,T)\times(0,L))},
\end{aligned}$$

where $C_1 = C_\epsilon(E, \rho, \kappa, \alpha, \frac{1}{\beta})(1 + k + e^k k^2)$, $k = \int_0^t r(s)ds$, $\bar{c_\delta}$ *is positive and* $r(t) = \|\nabla\boldsymbol{u}^1\|_2^2(t) + 1 + \|\partial_t h\|_\infty(t)ds + a\left\|\frac{\partial R_0}{\partial y_1}\right\|_{L^2(0,L)}(t)$ *and* $\omega(t) \to 0$ *if* $t \to 0$.

Note 5.1. *Let us point out that on the right hand side of (5.18) the difference* $\|h^1 - h^2\|_{W^{1,\infty}((0,T)\times(0,L))}$ *appears. This inequality is a tool to prove the convergence of the iterative process with respect to the domain deformation* $h = R_0 + \eta^k$. *In fact, it is possible to show (at least under some assumptions) that the mapping* $A[\eta^k] = \eta^{k+1}$ *is contractive. The proof of convergence is shown in [9, Section 10].*

6 Original problem, $\varepsilon \to 0,\ \kappa \to \infty$.

The aim of this section is to prove the convergence of weak solution dependent on regularization parameters $\varepsilon,\ \kappa$ to the weak solution of original problem (1.1)–(1.11). We point out the dependence of weak solution on the parameters in the following way $\boldsymbol{u}_\varepsilon,\ q_\varepsilon,\ u_\varepsilon$. Since the strong convergence argument for $\varepsilon \to 0$ differs from argument for $\kappa \to \infty$, we will show this convergence in two steps. First we let $\varepsilon \longrightarrow 0$, keeping κ fixed and then we let $\kappa \longrightarrow \infty$.

Analogously as in Section 4.1 we obtain continuous form of the first a priori estimate by testing (2.6) with $(\boldsymbol{u}_\varepsilon, q_\varepsilon, \lambda u_\varepsilon + (1-\lambda)\partial_t h)$ and using property (4.51).

$$\begin{aligned}
&\int_D h(t)\left(|\boldsymbol{u}_\varepsilon|^2 + \varepsilon|q_\varepsilon|^2\right)(t)dy + \frac{\lambda E}{\rho}\int_0^L |u_\varepsilon(t)|^2 dy_1 \\
&+ \int_0^T \int_D \delta|\nabla \boldsymbol{u}_\varepsilon(t)|^p + \frac{2\alpha\varepsilon}{2+K^2}|\nabla q_\varepsilon(t)|^2 dy\,dt + \int_0^T\int_0^L \frac{\lambda c E}{\rho}\left|\frac{\partial u_\varepsilon}{\partial y_1}\right|^2 dy_1\,dt \\
&+ \int_0^L \frac{aE\lambda}{2\rho}\left|\int_0^t \frac{\partial u_\varepsilon(s)}{\partial y_1}ds\right|^2 + \frac{bE\lambda}{2\rho}\left|\int_0^t u_\varepsilon(s)s\right|^2 dy_1 \\
&+ \int_0^T\int_0^L \frac{2\kappa}{\rho}\left|\lambda u_\varepsilon + (1-\lambda)\partial_t h - u_{\varepsilon\,2}\right|^2 dy_1\,dt \qquad (6.1)\\
&\le \int_0^T M\|q_{\partial D}\|_{L^q(\partial D)} + c_1\left\|\frac{\partial^2 R_0}{\partial y_1^2}\right\|_{L^2(0,L)} dt \\
&\quad + c_2(1-\lambda)\int_0^L \max_{0\le t\le T}\left\{a\left|\partial_{y_1} h\right|^2 + b|h|^2 + 4|\partial_t h|^2\right\} dy_1 \\
&\quad + c_3(1-\lambda)\int_0^T\int_0^L \left\{\left|\frac{\partial^2 h}{\partial t^2}\right|^2 + c^2\left|\frac{\partial^2 h}{\partial t\partial y_1}\right|^2 + a^2|\partial_{y_1}h|^2 + b^2|h|^2\right\} dy_1\,dt
\end{aligned}$$

Note that the right hand side is independent on ε, κ.

6.1 Limiting process $\varepsilon \to 0$

In this subsection we keep parameter κ fixed. The estimate (6.1) implies the weak convergence of

$$(\boldsymbol{u}_\varepsilon, \sqrt{\varepsilon}q_\varepsilon, u_\varepsilon) \rightharpoonup (\boldsymbol{u}, \tilde{q}, u) \text{ in } L^p(0,T;\boldsymbol{V})\times L^2(0,T;H^1(D))\times L^2(0,T;H^1(0,L)) \qquad (6.2)$$

as $\varepsilon \to 0$. Moreover, after inserting test functions $(\mathbf{0}, \phi, 0)$ into (2.6) for sufficiently smooth ϕ we obtain

$$\int_0^T\int_D h\phi \operatorname{div}\boldsymbol{u} \le \sqrt{\varepsilon}C\|\sqrt{\varepsilon}q\|_{L^2(0,T;H^1(D))}\|\phi\|_{L^2(0,T;H^1(D))}. \qquad (6.3)$$

Using boundedness of $\sqrt{\varepsilon}q$ in $L^2(0,T;H^1(D))$ and tending ε to zero we have

$$\mathrm{div}_h \boldsymbol{u} = 0 \qquad a.e. \quad \text{on} \quad D \times (0,T).$$

This fact allows us to constrict the space of test functions to solenoidal test functions, i.e. $\mathrm{div}_h \boldsymbol{\psi} = 0$ a.e. on D.

Now, in order to get strong convergences for $h\boldsymbol{u}_\varepsilon$, we need a priori estimates concerning (weak distributive) time derivatives. Analogously as in Section 4.1 we test (2.6) with $(\boldsymbol{\psi}, 0, 0)$ such that $\mathrm{div}_h \boldsymbol{\psi} = 0$ and we get

$$\int_0^T \left\langle \frac{\partial (h\boldsymbol{u}_\varepsilon)}{\partial t}, \boldsymbol{\psi} \right\rangle \leq C.$$

This estimate is independent on ε, since the term $\int_0^T \int_D hq \mathrm{div}_h \boldsymbol{\psi}$ is canceled due to solenoidal test functions. Therefore we have

$$\frac{\partial (h\boldsymbol{u}_\varepsilon)}{\partial t} \in L^q(0,T;\mathcal{W}^*),$$

q is given by (4.19) and $\mathcal{W} = \{f \in \boldsymbol{V}, \mathrm{div}_h f = 0 \ a.e. \text{ on } D\}$. The Lions-Aubin lemma implies then strong convergence of $h\boldsymbol{u}_\varepsilon$ in $L^p(0,T;\mathcal{W})$.

Now, we are prepared to let $\varepsilon \to 0$ in (2.6). We use the weak convergences of $\boldsymbol{u}_\varepsilon$ in $L^p(0,T;\mathcal{W})$, $\sqrt{\varepsilon}q_\varepsilon$ in $L^2(0,T;H^1(D))$, u_ε in $L^2(0,T;H_0^1(0,L))$, see (6.2), the Minty-Browder theorem for the viscous term and strong convergence of $h\boldsymbol{u}_\varepsilon$ for in $L^2((0,T)\times D)$. The reader can note, that the strong convergences of q_ε for $\varepsilon \to 0$ can not be obtained using the Lions-Aubin lemma, since

$$\int_0^T \left\langle \frac{\sqrt{\varepsilon}\partial (hq_\varepsilon)}{\partial t}, \phi \right\rangle_{H^1} \leq C\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

Nevertheless, this strong convergence is not needful for limiting process in (2.6). After the limiting process $(\boldsymbol{u}_\varepsilon, \sqrt{\varepsilon}q_\varepsilon, u_\varepsilon) \to (\boldsymbol{u}, \tilde{q}, u)$ we arrive at

$$\begin{aligned}
&\int_0^T \int_D \left\{ h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial (y_2 \boldsymbol{u})}{\partial y_2}\boldsymbol{\psi} \right\} dy\, dt \qquad (6.4)\\
&= \int_0^T \Bigg\{ \ ((\boldsymbol{u}, \boldsymbol{\psi})) \ + \ b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\psi}) \\
&\qquad + \int_0^1 h(L)q_{out}(y_2,t)\psi_1(L,y_2,t) - h(0)q_{in}(y_2,t)\psi_1(0,y_2,t)\, dy_2 \\
&\qquad + \int_0^L \left(q_w + \frac{1}{2}\frac{\partial h}{\partial t}u_2 \right)\psi_2(y_1,1,t)\, dy_1 \\
&\qquad + \kappa \int_0^L \left(u_2 - \lambda u - (1-\lambda)\frac{\partial h}{\partial t} \right)\left(\frac{\psi_2}{\rho}(y_1,1,t) - \frac{\xi}{E}(y_1,t) \right) dy_1 \\
&\qquad + \int_0^L \Bigg(-u\frac{\partial \xi}{\partial t} + c\frac{\partial u}{\partial y_1}\frac{\partial \xi}{\partial y_1} + a\frac{\partial}{\partial y_1}\int_0^t u(y_1,s)ds\frac{\partial \xi}{\partial y_1} \\
&\qquad\qquad - a\frac{\partial^2 R_0}{\partial y_1^2}\xi + b\int_0^t u(y_1,s)ds\xi \Bigg)(y_1,t)\, dy_1 \Bigg\}\, dt,
\end{aligned}$$

for every $\boldsymbol{\psi} \in L^p(0,T;\boldsymbol{V}) \cap H^1(0,T;L^2(D))$, $\xi \in L^2(0,T;H_0^1(0,L))) \cap H^1(0,T;L^2(0,L))$, such that $\text{div}_h \boldsymbol{\psi} = 0$ a.e. on D.

Thus, we have proved the following existence result.

Theorem 6.1.
Assume that $p > (1+\sqrt{5})/2$ and $h \in W^{1,\infty}((0,T)\times(0,L)) \cap W^{2,2}((0,T)\times(0,L))$ satisfy (1.12), (1.13). Further assume that properties (3.1)–(3.4) hold and that $q_{in}, q_{out} \in L^\infty(0,T;L^2(0,1))$, $q_w \in L^\infty(0,T;L^2(0,L))$.

Then there exists a weak solution $(\boldsymbol{u},u)$ of problem (1.1)-(1.11) such that
i) $(\boldsymbol{u},u) \in [L^p(0,T;\boldsymbol{V}) \times L^2(0,T;H_0^1(0,L))] \cap [L^\infty(0,T;L^2(D)) \times L^\infty(0,T;L^2(0,L))]$
ii) $\boldsymbol{u}$ satisfies the condition $\text{div}_h \boldsymbol{u} = 0$ a.e on D and the integral identity (6.4) holds.

Note 6.1 (Weak time distributive derivative)**.** *Since we have shown independently on ε*

$$\partial_t \boldsymbol{u}_\varepsilon \rightharpoonup \chi, \quad \chi \in L^q(0,T;\mathcal{W}^*), \quad \textit{for } q \textit{ given by } (4.19),$$

we would like to show that χ is a weak distributive derivative of $\boldsymbol{u}$, i.e. $\chi = \partial_t(h\boldsymbol{u}) \in L^q(0,T;\mathcal{W}^*)$. We start with analogous considerations as those in Section 4.2.1. and obtain (4.47) and (4.48). The difficulty lies in the choice of test function $\boldsymbol{\psi}(x,t) = \boldsymbol{w}(x)\xi(t)$ such that $\boldsymbol{w} \in W^{1,p}(D)$, div $\boldsymbol{w}_h = 0$, $\xi \in C_0^1(0,T)$. Since now the divergence-free condition depends on time (on $h(x_1,t)$), the function $\boldsymbol{w}$ depends implicitly also on time. Here we must stop our further consideration concluding that we are only able to show that

$$\int_0^T \left\langle \chi, \boldsymbol{\psi} \right\rangle_{\mathcal{W}} = -\int_0^T \left\langle h\boldsymbol{u}, \frac{\partial \boldsymbol{\psi}}{\partial t} \right\rangle_{\mathcal{W}}.$$

Since we did not show that $\chi = \frac{\partial \boldsymbol{u}}{\partial t}$, we cannot show the property of weak time derivative (4.51) and we will not study uniqueness.

6.2 Limiting process $\kappa \to \infty$

As pointed out before using the same technique as in Section 4.1 we get estimates of time derivatives $\partial_t \boldsymbol{u}_\kappa$, $\partial_t u_\kappa$ (4.13), (4.19), which depends on κ. Therefore in the limiting process for $\kappa \to \infty$ we cannot use the Lions-Aubin lemma as in Lemma 4.2 in order to obtain strong convergences in appropriate spaces for $(\boldsymbol{u}_\kappa, u_\kappa) \to (\boldsymbol{u}, u)$.

In fact, we have to use another argument to obtain the strong convergence. We follow the lines of [9, Section 8], cf. also [2, Alt,Luckhaus, Lemma 1.9], and show the **third a priori estimate:**

$$\int_0^{T-\tau} \int_D |(h\boldsymbol{u}_{\varepsilon,\kappa})(t+\tau) - (h\boldsymbol{u}_{\varepsilon,\kappa})(t)|^2 + \varepsilon|(hq_{\varepsilon,\kappa})(t+\tau) - (hq_{\varepsilon,\kappa})(t)|^2 dy dt$$
$$+ \int_0^{T-\tau} \int_0^L |(hu_{\varepsilon,\kappa})(t+\tau) - (hu_{\varepsilon,\kappa})(t)|^2 dy_1 dt \le C\tau, \tag{6.5}$$

where C is positive constant independent on $\tau, \kappa, \varepsilon$. To obtain (6.5) we turn back to the previous case with fixed ε and test (2.6) with separable test functions $(\chi^\delta \boldsymbol{w}, \chi^\delta p, \chi^\delta \frac{E}{\rho} v)$, where $\chi^\delta(t)$ is smooth approximation of characteristic function of interval $(t, t+\tau)$ and $(\boldsymbol{w}(y), p(y), v(y_1)) \in V$. The reader can note that in the case of $\varepsilon = 0$ we would not be able to chose separable test function $\boldsymbol{\psi} = \xi(t)\boldsymbol{w}(y)$, since condition $\mathrm{div}_h \boldsymbol{\psi} = 0$ would imply the dependence of $w(y)$ on time. Then we put

$$\boldsymbol{w}(y) = \partial_t^\tau (h\boldsymbol{u}_{\varepsilon,\kappa}), \quad p(y) = \partial_t^\tau (hq_{\varepsilon,\kappa}), \quad u(y_1) = \partial_t^\tau \left(\lambda h u_{\varepsilon,\kappa} + (1-\lambda) h \partial_t h\right),$$

where $\partial_t^\tau f := f(t+\tau) - f(t)$ and integrate with respect to t over $(0, T-\tau)$. We arrive at

$$\begin{aligned}
&\int_0^{T-\tau} \int_D |\partial_t^\tau (h\boldsymbol{u}_{\varepsilon,\kappa})|^2 + |\partial_t^\tau (hq_{\varepsilon,\kappa})|^2 + \lambda \frac{E}{\rho} \int_0^{T-\tau} \int_0^L h |\partial_t^\tau (u_{\varepsilon,\kappa})|^2 dy_1 dt \\
&= -\frac{E}{\rho} \int_0^{T-\tau} \int_0^L \lambda u_{\varepsilon,\kappa} \partial_t^\tau (u_{\varepsilon,\kappa}) \partial_t^\tau h + (1-\lambda) \partial_t^\tau (u_{\varepsilon,\kappa}) \partial_t^\tau (h \partial_t h) \qquad (6.6) \\
&+ \int_0^{T-\tau} \int_t^{t+\tau} \int_D ((\boldsymbol{u}_{\varepsilon,\kappa}(s), \partial_t^\tau h \boldsymbol{u}_{\varepsilon,\kappa})) - h(s) q_{\varepsilon,\kappa}(s) ds\, \mathrm{div} \partial_t^\tau h \boldsymbol{u}_{\varepsilon,\kappa} \\
&\qquad + h(s) \mathrm{div} \boldsymbol{u}(s) \partial_t^\tau h q_{\varepsilon,\kappa} + \ldots dy + \int_0^L \ldots dy_1 + \int_0^1 \ldots dy_2 ds dt.
\end{aligned}$$

In order to see, that the right hand side of (6.6) is bounded with $C\tau$ from above and does not depend on ε, κ we refer to property (6.3) and also to [9, Section 8]. Here it is proven that the corresponding boundary term with κ:

$$\begin{aligned}
&\kappa\tau \int_0^{T-\tau} \int_0^L \left[u_2 - \lambda u - (1-\lambda)\partial_t h\right]_\tau (t) \qquad (6.7) \\
&\qquad \times \partial_t^\tau \left(h \Big(u_2 - \lambda u - (1-\lambda) \partial_t h \Big) \right) dy_1 dt \le C\tau
\end{aligned}$$

is bounded on the base of (6.1) independently on κ. Denoting the so-called Steklov average

$$[\phi]_\tau (t = \frac{1}{\tau} \int_t^{t+\tau} \phi(s) ds.$$

one can also shown that $\|[\phi]_\tau\|_{L^2((0,T-\tau)\times D)} \le \|\phi\|_{L^2((0,T)\times D)}$, which is used in (6.7).

We pay our attention now to the new viscous term $((\boldsymbol{u}_{\varepsilon,\kappa}(s), \partial_t^\tau h \boldsymbol{u}_{\varepsilon,\kappa}))$ on the right hand side of (6.6) and show, that it is bounded with $C\tau$. Indeed, we get (written without indices ε, κ)

$$\begin{aligned}
&\tau \int_0^{T-\tau} \int_D \frac{1}{\tau} \int_t^{t+\tau} \tau_{ij} \big(\hat{e}(\boldsymbol{u}(s))\big) ds \big\{ \hat{e}_{ij}(h\boldsymbol{u}(t+\tau)) - \hat{e}_{ij}(h\boldsymbol{u}(t)) \big\} dy dt \\
&\overset{(5.13)}{\le} C_5 \tau \int_0^{T-\tau} \int_D \big[1 + |\hat{e}(\boldsymbol{u})|^{p-1}\big]_\tau (t) \big\{ \hat{e}_{ij}(h\boldsymbol{u}(t+\tau)) - \hat{e}_{ij}(h\boldsymbol{u}(t)) \big\} dy dt \\
&\overset{Holder}{\le} c\tau \big\| [1 + |\hat{e}(\boldsymbol{u})|^{p-1}]_\tau (t) \big\|_{L^q} \Big\| |\hat{e}(\boldsymbol{u}(t+\tau))| + |\hat{e}(\boldsymbol{u}(t))| \Big\|_{L^p}, \qquad (6.8)
\end{aligned}$$

where $q = p/(p-1)$ is the dual index to p, and we have denoted the norm $\|\cdot\|_{L^p((0,T-\tau)\times D)}$ by $\|\cdot\|_{L^p}$. Analogously as above we obtain

$$\|[\phi]_\tau\|_{L^q((0,T-\tau)\times D)} \leq \|\phi\|_{L^q((0,T)\times D)} \quad \forall q > 1. \tag{6.9}$$

Since $\||\boldsymbol{u}(t)|^{p-1}\|_{L^q((0,T)\times D)} = \|\boldsymbol{u}(t)\|^{p-1}_{L^p((0,T)\times D)}$ we conclude from (6.8) and (6.9) that

$$\int_0^{T-\tau}\int_D\int_t^{t+\tau} ((\boldsymbol{u}_{\varepsilon,\kappa}(s), \partial_t^\tau h\boldsymbol{u}_{\varepsilon,\kappa}))\, ds\, dy\, dt \leq c\tau.$$

Estimates of other terms on the right hand side of (6.6) has been done in [26] and [9].

On the base of a priori estimate (6.6) and due to the compactness argument from [2, Lemma 1.9] we get for $\kappa \to \infty$ the following strong convergences

$$\boldsymbol{u}_\kappa \to \boldsymbol{u} \;\text{ in } L^r((0,T)\times D), \qquad u_\kappa \to u \;\text{ in } L^s((0,T)\times(0,L)),$$

where $1 \leq r < 7/3,\ 1 \leq s < 6$ for $\kappa \to \infty$, cf. (4.37) and [9, Section 8]. Now take a special test function $\xi(y_1,t) = \frac{E}{\rho}\boldsymbol{\psi}_2(y_1,1,t)$ and let $\kappa \to \infty$ in (6.4), (written now with $\boldsymbol{u}_\kappa$ instead of $\boldsymbol{u}$). The limiting process leads to

$$\begin{aligned}
&\int_0^T\int_D \left\{ h\boldsymbol{u}\cdot\frac{\partial\boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial(y_2\boldsymbol{u})}{\partial y_2}\cdot\boldsymbol{\psi}\right\} dy\, dt \qquad (6.10)\\
&= \int_0^T \Bigg\{ \; ((\boldsymbol{u},\boldsymbol{\psi}))_h \; + \; b_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi})\\
&\qquad \int_0^1 h(L)q_{out}(y_2,t)\psi_1\,(L,y_2,t) - h(0)q_{in}(y_2,t)\psi_1\,(0,y_2,t)\, dy_2\\
&\qquad + \int_0^L \left(q_w + \frac{1}{2}\frac{\partial h}{\partial t}u_2\right)\psi_2\,(y_1,1,t)\, dy_1\\
&\qquad + \int_0^L \Bigg(-u\frac{\partial\xi}{\partial t} + c\,\frac{\partial u}{\partial y_1}\frac{\partial\xi}{\partial y_1} + a\frac{\partial}{\partial y_1}\int_0^t u(y_1,s)ds\frac{\partial\xi}{\partial y_1}\\
&\qquad\qquad -a\frac{\partial^2 R_0}{\partial y_1^2}\xi + b\int_0^t u(y_1,s)ds\xi\Bigg)\,(y_1,t)\, dy_1 \Bigg\}\, dt\;.
\end{aligned}$$

Using conditions (1.16), (1.17) we obtain the original Dirichlet boundary condition $\partial_t\eta = u_2$ for $\lambda = 1$.

Finally we transform (6.10) to the original moving domain $\Omega(h)$ and conclude the paper with a main result on the existence of a weak solution to our original problem (1.1)-(1.11) with a prescribed function of domain deformation.

Theorem 6.2 (Existence of original problem)**.**
Let $\lambda = 1$ and $p > (1+\sqrt{5})/2$. Assume that $h \in W^{1,\infty}((0,T)\times(0,L))$ satisfies (1.12), (1.13) and that the boundary data fulfill $q_{in}, q_{out} \in L^\infty(0,T;L^2(0,1))$,

$q_w \in L^\infty(0,T;L^2(0,L))$. Furthermore, assume that the properties (3.1)–(3.4) for the viscous stress tensor hold.

Then there exists a weak solution $(\boldsymbol{v},\eta)$ of the problem (1.1)-(1.11) such that
i) $(\boldsymbol{u},u) \in [L^p(0,T;\boldsymbol{V})\times L^2(0,T;H_0^1(0,L))]\cap[L^\infty(0,T;L^2(D))\times L^\infty(0,T;L^2(0,L))]$, where $(\boldsymbol{u},u)$ is defined in (2.1)
ii) $\boldsymbol{v}$ satisfies the divergence free condition div $\boldsymbol{v}=0$ a.e on $\Omega(h)$ and the following integral identity holds

$$\begin{aligned}
&\int_{\Omega(h)}\left\{-\boldsymbol{v}\cdot\frac{\partial\varphi}{\partial t}+div\left[\frac{2}{\rho}\mu(|e(\boldsymbol{v})|)e(\boldsymbol{v})\right]+\sum_{i,j=1}^{2}v_i\frac{\partial v_j}{\partial x_i}\varphi_j\right\}dxdt\\
&+\int_0^T\int_{\Gamma_{out}}\left(P_{out}-\frac{1}{2}|v_1|^2\right)\varphi_1(L,x_2,t)\,dSxdt\\
&-\int_0^T\int_{\Gamma_{in}}\left(P_{in}-\frac{1}{2}|v_1|^2\right)\varphi_1(0,x_2,t)\,dSxdt\\
&+\int_0^T\int_{\Gamma_w}\left(P_w-\frac{1}{2}v_2\left(v_2-\frac{\partial h}{\partial t}\right)\right)\varphi_2(x_1,h(x_1,t),t)\,dSxdt\\
&+\int_0^T\int_0^L\left(-\frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial t}+c\,\frac{\partial^2\eta}{\partial x_1\partial t}\frac{\partial\xi}{\partial x_1}+a\frac{\partial\eta}{\partial x_1}\frac{\partial\xi}{\partial x_1}+b\eta\,\xi\right)(x_1,t)\,dx_1dt\;=\;0
\end{aligned}$$

for every test functions

$$\varphi=(x_1,x_2,t)=\boldsymbol{\psi}\left(x_1,\frac{x_2}{h(x_1,t)},t\right),\quad \boldsymbol{\psi}\in L^p(0,T;\boldsymbol{V})\cap H^1(0,T;L^2(D)),$$
$$\textit{such that div } \varphi=0 \quad \textit{a.e on } \Omega(h)$$
$$z=(x_1,t)=\int_0^t\xi(x_1,s)ds,\;\;\xi\in L^2(0,T;H_0^1(0,L))\cap H^1(0,T;L^2(0,L)).$$

Note that the structure equation is fulfilled in a slightly modified sense

$$E\left[\frac{\partial^2\eta}{\partial t^2}-a\frac{\partial^2\eta}{\partial x_1^2}+b\eta-c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right](x_1,t)=$$
$$-\mathbf{T}_f\boldsymbol{n}\cdot\boldsymbol{e}_r-P_w\mathbf{I}\boldsymbol{n}\cdot\boldsymbol{e}_r+\frac{\rho}{2}v_2(v_2-\partial_t h)+aE\frac{\partial^2R_0}{\partial x_1^2}$$

a.e. on $[0,L]\times(0,T)$.

Moreover, the interface boundary condition holds at the moving wall Γ_w

$$v_2=\frac{\partial\eta}{\partial t}\qquad \text{a.e on}\;\;\Gamma_w, t\in(0,T).$$

7 Conclusions

In the present paper we have generalized the result obtained in [9] for the Newtonian fluids to the shear dependent non-Newtonian fluids. The non-linear stress tensor satisfies the polynomial growth conditions (3.1)–(3.4).

This allows us using the energy method and monotonicity arguments based on the Minty-Browder theorem to study the existence and uniqueness of the $(\kappa,\ \varepsilon)$ approximate solutions (2.6). For $p > (1+\sqrt{5})/2$ the existence of weak solutions has been shown also for limiting case $\kappa \to \infty$ and $\varepsilon \to 0$. The uniqueness and continuous dependence on data are obtained only for the $\kappa,\ \varepsilon$-approximate weak solutions in case $p \geq 2$.

The results presented here as well as in [9] are obtained for a moving domain $\Omega(h)$ with the prescribed function of domain deformation.

It remains to show that the mapping $A[\eta^{(k)}] = \eta^{(k+1)}$ is contractive, cf. Note 5.1. As it is pointed out the proof of convergence of the iterative process with respect to the domain deformation has been illustrated for a special case in [9]. In future we want to study this question more deeply using, for example, techniques allowing to work with divergence free functions in moving domains, see [10, 4, 5].

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