STRATEGIES FOR IMPROVING AUTOMATED MULTI-LEVEL SUB-STRUCTURING

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Abstract

Frequency response analysis for complex dynamical systems requires the solution of huge eigenvalue problems. The Automated Multi-Level Sub-structuring (AMLS) method is a multi-level extension of the component mode synthesis method which has recently be shown to be efficient for performing vibration analysis for large scale models, in particular if a large number of eigenpairs is needed. On each level the stiffness matrix of substructures are decoupled from the remaining degrees of freedom, and neglecting the coupling in the mass matrix the size of the substructure is reduced by modal truncation where eigen information corresponding to frequencies which exceed a given threshold is chopped off. In this paper we discuss two alternative reduction methods based on moment matching and on Krylov subspaces which both incorporate the coupling of sub-structures and their boundaries, and we evaluate their approximation properties for a two-level version.

Keywords: automated multi-level sub-structuring, AMLS, eigenvalue, eigenvector, sparse matrix, Krylov subspace method, moment matching

1 Introduction

The numerical solution of huge sparse symmetric eigenvalue problems is one of the most challenging tasks in numerical linear algebra. Iterative projection methods such as the Lanczos algorithm (combined with block variants and shift-and-invert techniques) or the Jacobi–Davidson method have dominated the scene

Over the last few years, a new method for huge eigenvalue problems, known as *Automated Multi–Level Sub-structuring (AMLS)*, has been developed by Bennighof and co-authors, and has been applied to frequency response analysis of complex structures [1, 2]. Here the large finite element model is recursively divided into very many substructures on several levels based on the sparsity structure of the system matrices.

Assuming that the interior degrees of freedom of substructures depend quasi-statically on the interface degrees of freedom, and modelling the deviation from quasi-static dependence in terms of a small number of selected substructure eigenmodes, the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest. Recent studies ([3, 4], e.g.) in vibro-acoustic analysis of passenger car bodies where huge FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed have shown that AMLS is considerably faster than Lanczos type approaches for this sort of problems.

It is a drawback of the standard form of AMLS that only eigenmodes of the clamped sub-structures are included into the search space the eigenfrequencies of which do not exceed a given cut-off frequency. This approach does not incorporate the coupling of substructures and their boundaries sufficiently. An alternative selection rule of sub-structure eigenmodes that takes into account the force applied by the sub-structure to its boundary was presented by Bai, Liao and Gao [5, 6]. A further possibility to include the interaction of a sub-structure and its boundary is to replace eigenmodes by bases of block-Krylov subspaces which allow for the interaction via the initial block. For the special symmetric eigenvalue problem $Ax = \lambda x$ this method was proposed by Bekas and Saad [7] (although motivated differently). In this paper we discuss both alternative reduction methods (which were tested numerically only for single-level examples), and we compare their approximation properties with the standard AMLS method for a two-level version.

Our paper is organised as follows: Section 2 briefly summarises the automated multi-level sub-structuring method for generalised linear eigenvalue problems. Section 3 presents the modified mode selection rule, and Section 4 derives the block-Krylov method, and expounds the problems of using the method in a multi-level environment. Section 5 reports on numerical experiments for a multi-level implementation of the new AMLS versions. These demonstrate that the modified mode selection approach is promising. The AMLS method based on block-Krylov subspaces seems to be inferior to the standard AMLS algorithm, and more research is necessary to make it a competitive method for large scale problems.

2 Automated Multi-Level Sub-structuring

In this section we review the basic ideas behind AMLS for solving the linear eigenvalue problem

$$Kx = \lambda Mx \tag{1}$$

where K and M are real, symmetric $n \times n$ matrices, M is positive definite, and both matrices are sparse.

We first consider the single-level version of the AMLS method. Assume that the

rows and columns of K and M are permuted such that

$$K = \begin{pmatrix} K_{11} & & K_{31}^T \\ & K_{22} & K_{32}^T \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \quad \text{end} \quad M = \begin{pmatrix} M_{11} & & M_{31}^T \\ & M_{22} & M_{32}^T \\ M_{31} & M_{32} & M_{33} \end{pmatrix}, \tag{2}$$

where $K_{11}, M_{11} \in \mathbb{R}^{n_1 \times n_1}$ and $K_{22}, M_{22} \in \mathbb{R}^{n_2 \times n_2}$ correspond to two substructures, $K_{33}, M_{33} \in \mathbb{R}^{n_3 \times n_3}$ describes the interface between these substructures, and K_{3i}, M_{3i}, $i = 1, 2$ indicates the coupling of substructure i to the interface.

To remove the off-diagonal blocks K_{3i}, $i = 1, 2$ in K we apply the congruent transformation with

$$T^T K T y = \lambda T^T M T y, \quad y = T^{-1} x$$

where

$$T = \begin{pmatrix} I & O & -K_{11}^{-1} K_{31}^T \\ O & I & -K_{22}^{-1} K_{32}^T \\ O & O & I \end{pmatrix}. \tag{3}$$

Then problem (1) is equivalent to

$$\tilde{K} y := \begin{pmatrix} K_{11} & O & O \\ O & K_{22} & O \\ O & O & \tilde{K}_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \lambda \begin{pmatrix} M_{11} & O & \tilde{M}_{31}^T \\ O & M_{22} & \tilde{M}_{32}^T \\ \tilde{M}_{31} & \tilde{M}_{32} & \tilde{M}_{33} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} =: \lambda \tilde{M} y, \tag{4}$$

where

$$\begin{aligned} \tilde{K}_{33} &= K_{33} - \sum_{i=1,2} K_{3i} K_{ii}^{-1} K_{3i}^T, \\ \tilde{M}_{33} &= M_{33} - \sum_{i=1,2} (M_{3i} K_{ii}^{-1} K_{3i}^T - K_{3i} K_{ii}^{-1} M_{3i}^T - K_{3i} K_{ii}^{-1} M_{ii} K_{ii}^{-1} K_{3i}^T), \\ \tilde{M}_{3i} &= M_{3i} - K_{3i} M_{ii} K_{ii}^{-1}, \ i = 1, 2. \end{aligned}$$

Notice that problems (1) and (4) are equivalent, i.e. the eigenvalues are identical and if y is an eigenvector of (4) then $x = Ty$ is an eigenvector of (2).

Let the columns of $S_j \in \mathbb{R}^{n_j \times k_j}$, $j = 1, 2$ consist of k_j selected eigenvectors of the eigenproblems $K_{jj} z = \omega M_{jj} z$ normalised such that $S_j^T M_{jj} S_j = I$ (which are called substructure eigenmodes), and let the columns of S_3 consist of k_3 selected eigenvectors of $\tilde{K}_{33} z = \omega \tilde{M}_{33} z$ with $S_3^T \tilde{M}_{33} S_3 = I$ (called interface eigenmodes). The single level reduction is obtained by projecting problem (4) to the subspace spanned by the columns of

$$S = \begin{pmatrix} S_1 & & \\ & S_2 & \\ & & S_3 \end{pmatrix} \in \mathbb{R}^{n \times k}, \ k := k_1 + k_2 + k_3. \tag{5}$$

Approximation of eigenvalues of (1) are obtained from the projected problem

$$S^T \tilde{K} S w = \theta S^T \tilde{M} S w, \tag{6}$$

which obtains the following form

$$\hat{K}z := \begin{pmatrix} \Omega_1 & O & O \\ O & \Omega_2 & O \\ O & O & \Omega_3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \theta \begin{pmatrix} I & O & \hat{M}_{31}^T \\ O & I & \hat{M}_{32}^T \\ \hat{M}_{31} & \hat{M}_{32} & \hat{M}_{33} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} =: \lambda \hat{M} z. \qquad (7)$$

Here Ω_j are diagonal matrices containing the retained eigenvalues of the substructure problems and the interface problem, and $\hat{M}_{3j} = S_3^T \tilde{M}_{3j} S_j$, $j = 1, 2, 3$.

It follows from the minmax characterisation of eigenvalues that the eigenvalues $\theta_1 \leq \theta_2 \leq \ldots$ of (7) ordered by magnitude are upper bounds of the corresponding eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ of the original problem. They are the best eigenvalue approximations of (1) from the subspace span$\{S\}$.

This approach contains classical methods from structural analysis: for $k_1 = k_2 = 0$ and S_3 replaced by the identity I one obtains static condensation or Guyan reduction [8]

$$\tilde{K}_{33} z_3 = \lambda \tilde{M}_{33} z_3, \qquad (8)$$

and $k_1, k_2 > 0$ yields the component mode synthesis method introduced by Craig and Bampton [9].

The selection of eigenmodes of substructures should be made such that the dimension k of the resulting projected eigenproblem (7) is small while a sufficient amount of spectral information of (1) is retained. Usually those eigenmodes are kept for which the corresponding eigenvalue does not exceed a given cut-off frequency. For this method an a priori bound was proved in [10] which however is very conservative and overestimates the error by one or two magnitudes. Liao, Bai and Gao [6] suggested an alternative mode selection procedure which is based on moment matching considerations and which sometimes yields better results than the methods based on cut-off frequencies. This method is presented in the Section 3. More research is necessary to develop more efficient strategies for mode selection.

The single-level sub-structuring method can be generalised to a multi-level version in a natural way. The same procedure that was used to partition K and M can be applied to (K_{11}, M_{11}) and (K_{22}, M_{22}) to generate a two-level sub-structuring. The matrix K then obtains the following form

$$K = \begin{pmatrix} K_{11} & & & & \text{sym.} & & \\ & K_{22} & & & & & \\ K_{31} & K_{32} & K_{33} & & & & \\ & & & K_{44} & & & \\ & & & & K_{55} & & \\ & & & K_{64} & K_{65} & K_{66} & \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} \end{pmatrix} \qquad (9)$$

and the matrix M receives the same structure. Repeating this sub-structuring for diagonal elements of K and M recursively one finally obtains the multi-level sub-structured form of the problem.

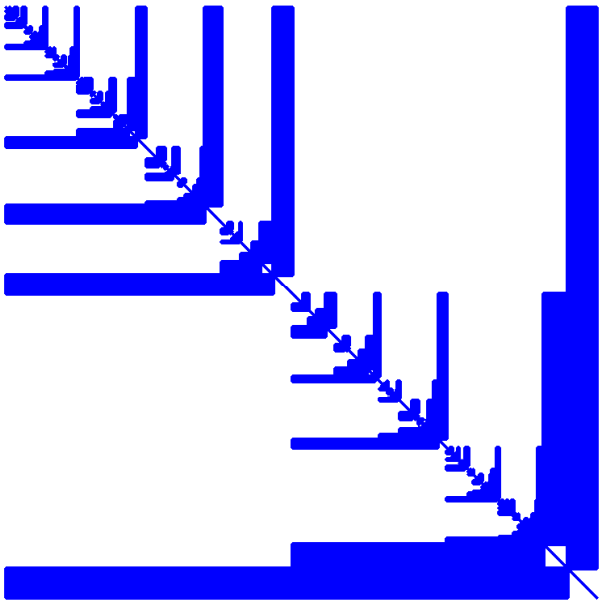

Figure 1: Transformed mass matrix.

In a way we explained a top down approach of AMLS. In implementations the bottom up approach is more appropriate [11]. One first determines a nested sub-structuring by a graph partitioning software such as METIS [12], and then reduces the constituent eigenvalue problems beginning with the leafs of the substructure tree. Addressing the substructures on coarser level in the same way, one finally arrives at an eigenvalue problem where the stiffness matrix has become diagonal, and the mass matrix obtains a generalised arrowhead structure as shown in Figure 1 (which generalises the arrowhead structure of the mass matrix in (7)).

3 Mode selection based on moment matching

A key step in sub-structuring methods is to compute and retain the modes of sub-systems. A standard mode selection practice is to retain the modes associated with few lowest eigenvalues, i.e. eigenvalues which do not exceed a predetermined cut-off frequency. Rules of thump like to keep those substructure eigenmodes for which the eigenvalue is at most twice or ten times as large as the largest wanted eigenvalue. But this is just heuristic and does not guarantee a good reduced eigenproblem.

In recent papers Bai, Gao and Liao [5, 6] proposed a different selection procedure for the single level sub-structuring which is based on moment matching from model order reduction. Consider the second order dynamical system

$$\begin{aligned} M\ddot{x}(t) + Kx(t) &= Bu(t) \\ y(t) &= L^T x(x) \end{aligned} \tag{10}$$

with initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$. t is the time variable, $x(t)$ the state vector, $u(t)$ the input excitation force vector, and $y(t)$ the output measurement vector.

Assume that K and M are sub-structured as in (2), and let the matrix S be given in (5). Projecting (10) to the subspace span$\{S\}$ with S_3 replaced by I yields the linear system

$$\begin{aligned} \hat{M}\ddot{z}(t) + \hat{K}z(t) &= \hat{B}u(t) \\ \hat{y}(t) &= \hat{L}^T z(t) \end{aligned} \tag{11}$$

where $\hat{K} = S^T K S$ and $\hat{M} = S^T M S$ are given in (7), $\hat{B} = S^T B$ and $\hat{L} = S^T L$, and in modal coordinates in frequency domain it is of the form

$$\left(-\omega^2 \begin{pmatrix} I & O & \hat{M}_{31}^T \\ O & I & \hat{M}_{32}^T \\ \hat{M}_{31} & \hat{M}_{32} & \tilde{M}_{33} \end{pmatrix} + \begin{pmatrix} \Omega_1 & O & O \\ O & \Omega_2 & O \\ O & O & \tilde{K}_{33} \end{pmatrix} \right) \begin{pmatrix} Z_1(\omega) \\ Z_2(\omega) \\ Z_3(\omega) \end{pmatrix} = \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \end{pmatrix} U(\omega) \tag{12}$$

We assume that no exterior force is acting on the substructures (i.e. $\hat{B}_1 = 0$, $\hat{B}_2 = 0$) but only on the interface. Solving the first two equations for Z_1 and Z_2 and substituting into the interface equation yields

$$\left(\omega^4 \sum_{i=1}^{2} (-\hat{M}_{3i}(-\omega^2 I + \Omega_i)^{-1} \hat{M}_{3i}^T) - \omega^2 \tilde{M}_{33} + \tilde{K}_{33} \right) Z_3(\omega) = \hat{B}_3 U(\omega) \tag{13}$$

Equation (13) can be interpreted as the force applied to the interface by the subsystems.

Let the power series of $Z_3(\omega)$ be denoted by

$$Z_3(\omega) = \Big(\sum_{j=0}^{\infty} r_j \omega^{2j} \Big) U(\omega).$$

Equating powers of ω^2 on both sides of (13) one easily obtains the first two moments r_0 and r_1 of $Z_3(\omega)$. The next moment r_2 is not that easily obtained because all substructure eigenmodes $\omega_j^{(i)}$ are involved. However, Bai, Liao and Gao [5, 6] derived the following selection rule : Choosing the modes $s_j^{(i)}$, $j = 1, \ldots, \ell$ of substructure i such that

$$\left\| \frac{1}{\omega_j^{(i)}} \hat{M}_{3i} s_j^{(i)} (s_j^{(i)})^T \hat{M}_{3i}^T \right\| \geq \left\| \frac{1}{\omega_k^{(i)}} \hat{M}_{3i} s_k^{(i)} (s_k^{(i)})^T \hat{M}_{3i}^T \right\|, \quad j = 1, \ldots, \ell,\ k > \ell \tag{14}$$

then the force applied to the interface by the substructure i is (in a certain sense) optimally represented for all selections of ℓ substructure modes.

It is interesting to note that the matrices $\hat{M}_{3i}$ which describe the coupling of substructure i to the interface are considered when selecting the retained substructure modes whereas the coupling is ignored in the selection rule of the component mode synthesis method.

Examples have shown (cf. [5, 6]) that for CMS (the single level version of AMLS) the mode selection method based on the rule (14) can be more accurate than the one based on a cut-off strategy. However, since the modes to be retained may not be in the natural order it is not clear how to make sure that the appropriate modes have been chosen. The generalisation to multi-level AMLS is obvious.

4 Krylov subspace expansion of projection spaces

CMS (and likewise AMLS) assumes that the interior degrees of freedom of substructures depend quasi-statically on the interface degrees of freedom, and models the deviation from quasi-static dependence in terms of a small number of selected substructure eigenmodes. Choosing the retained substructure eigenmodes based on a cut-off frequency rule the coupling of substructures is not considered sufficiently, and the modified selection procedure in the last section tried to improve this drawback. A different approach to incorporate the coupling of substructure and interfaces into the ansatz space of the projection method uses block Krylov subspaces.

Consider the eigenvalue problem (1) in the following form

$$\begin{pmatrix} K_{\ell\ell} & K_{\ell i} \\ K_{i\ell} & K_{ii} \end{pmatrix} \begin{pmatrix} x_\ell \\ x_i \end{pmatrix} = \lambda \begin{pmatrix} M_{\ell\ell} & M_{\ell i} \\ M_{i\ell} & M_{ii} \end{pmatrix} \begin{pmatrix} x_\ell \\ x_i \end{pmatrix} \tag{15}$$

where the index ℓ collects all unknowns in the interior of substructures (indices 1 and 2 in the representation (2)), and the set with index i (index 3 in (2)) contains all interface degrees of freedom. Notice that $K_{\ell\ell}$ and $M_{\ell\ell}$ are block matrices, and therefore applying $K_{\ell\ell}^{-1}$ means that a couple of linear systems of small dimension have to be solved.

Solving the first block equation for x_ℓ one obtains

$$x_\ell = -(K_{\ell\ell} - \lambda M_{\ell\ell})^{-1}(K_{\ell i} - \lambda M_{\ell i})x_i =: P_e x_i. \tag{16}$$

Linearising (16) at $\lambda = 0$ one gets the approximate relation

$$x_\ell = -K_{\ell\ell}^{-1} K_{\ell i} x_i =: P_s x_i \tag{17}$$

between the local degrees of freedom x_ℓ and the interface degrees of freedom x_i. Projecting the eigenvalue problem (15) to the subspace

$$\left\{ \begin{pmatrix} P_s \\ I \end{pmatrix} x_i \,:\, x_i \in \mathbb{R}^i \right\},$$

one gets the statically condensed eigenproblem. If x_i is an eigenvector of the projected eigenproblem, then $x_\ell := -K_{\ell\ell}^{-1} M_{\ell\ell} x_i$ is chosen as the local part of an approximate eigenvector of (15).

Substituting x_ℓ from (16) into the second block equation of (15) yields the exactly condensed eigenvalue problem

$$\big(K_{ii} - (K_{i\ell} - \lambda M_{i\ell})(K_{\ell\ell} - \lambda M_{\ell\ell})^{-1}(K_{\ell i} - \lambda M_{\ell i})\big)x_i = \lambda M_{ii} x_i. \tag{18}$$

It is nearly equivalent to the original eigenproblem (15): every eigenvalue of problem (18) is also an eigenvalue of (15), and conversely, if an eigenvalue λ of the original problem (15) is not an eigenvalue of the substructure eigenproblem

$$K_{\ell\ell}x_\ell = \omega M_{\ell\ell}x_\ell \tag{19}$$

then it is also an eigenvalue of the nonlinear problem (18).

The exactly condensed problem has an interesting structure. Its eigenvalues can be characterised as minmax values of a Rayleigh functional p, and eigenvectors x_i are stationary vectors of p (cf. [13]). Hence, the Rayleigh functional can be used to get an improved eigenvalue approximation of (15) from an eigenvector approximation $\tilde{x}_i$ (for instance an eigenvector of the statically condensed problem (17)).

If $|\lambda|$ is less than the smallest eigenvalue of the substructure eigenproblem (19), then

$$(K_{\ell\ell} - \lambda M_{\ell\ell})^{-1} = (I - \lambda K_{\ell\ell}^{-1}M_{\ell\ell})K_{\ell\ell}^{-1} = \sum_{j=0}^{\infty} \lambda^j (K_{\ell\ell}^{-1}M_{\ell\ell})^j K_{\ell\ell}^{-1}. \tag{20}$$

Therefore, the exact relation between the interface component x_i and the local component x_ℓ of an eigenvector corresponding to an eigenvalue not exceeding the smallest eigenvalue of (19) reads

$$\begin{aligned} x_\ell &= -(K_{\ell\ell} - \lambda M_{\ell\ell})^{-1}(K_{\ell i} - \lambda M_{\ell i})x_i \\ &= -\sum_{j=0}^{\infty} \lambda^j (K_{\ell\ell}^{-1}M_{\ell\ell})^j K_{\ell\ell}^{-1}(K_{\ell i} - \lambda M_{\ell i})x_i \\ &= -\sum_{j=0}^{\infty} \lambda^j (K_{\ell\ell}^{-1}M_{\ell\ell})^j K_{\ell\ell}^{-1}K_{\ell i}x_i + \sum_{j=0}^{\infty} \lambda^{j+1} (K_{\ell\ell}^{-1}M_{\ell\ell})^j K_{\ell\ell}^{-1}M_{\ell i}x_i, \end{aligned} \tag{21}$$

Hence, x_ℓ can be approximated by truncations of these infinite sums, i.e. by elements of the block-Krylov subspace

$$\begin{aligned} &\mathcal{K}_m(K_{\ell\ell}^{-1}M_{\ell\ell}, K_{\ell\ell}^{-1}[K_{\ell i}, M_{\ell i}]x_i) \\ &\quad = \text{span}\{K_{\ell\ell}^{-1}[K_{\ell i}, M_{\ell i}]x_i, \ldots, (K_{\ell\ell}^{-1}M_{\ell\ell})^{m-1}K_{\ell\ell}^{-1}[K_{\ell i}, M_{\ell i}]x_i\}. \end{aligned} \tag{22}$$

This suggests the following single level version of the Krylov AMLS method for computing approximations of eigenvalues and eigenvectors of (15):

(i) compute the m_i smallest eigenvalues and corresponding eigenvectors $u_1^{(i)}, \ldots, u_{m_i}^{(i)}$ of the statically condensed problem (i.e. $\tilde{K}_{33}u^{(i)} = \omega\tilde{M}_{33}u^{(i)}$ with notations from (4))

(ii) Compute an orthonormal basis U_{m_ℓ} of the block Krylov space

$$\mathcal{K}_{m_\ell}(K_{\ell\ell}^{-1}M_{\ell\ell}, K_{\ell\ell}^{-1}[K_{\ell i}, M_{\ell i}]U_m^{(i)})$$

with $U_{m_i}^{(i)} = [u_1^{(i)}, \ldots, u_{m_i}^{(i)}]$

(iii) Compute the wanted eigenvalues $\tilde{\lambda}_j$ and eigenvectors z_j of the projected problem

$$Z^T T^T K T Z z = \lambda Z^T T^T M T Z z \tag{23}$$

where

$$Z := \begin{pmatrix} U_{m_\ell} & 0 \\ 0 & U_{m_i} \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} I & -K_{\ell\ell}^{-1} K_{\ell i} \\ 0 & I \end{pmatrix} \tag{24}$$

and determine eigenvector approximations $\tilde{x}_j = TZz_j$.

Hence, instead of approximating the substructure portions of eigenvectors of (15) by eigenvectors of substructures one approximates them by elements of the block Krylov subspace thus making use of the coupling between the interface and the substructures.

For the case of the special eigenvalue

$$Ax = \begin{pmatrix} A_{\ell\ell} & A_{\ell i} \\ A_{i\ell} & A_{ii} \end{pmatrix} \begin{pmatrix} x_\ell \\ x_i \end{pmatrix} = \lambda x \tag{25}$$

this approach was proposed by Bekas and Saad in [7]. Then $M_{\ell i} = M_{i\ell}^T = 0$, $M_{ii} = I$, and $M_{\ell\ell} = I$, and block Krylov space reduces to

$$\mathcal{K}_m(A_{\ell\ell}^{-1}, A_{\ell\ell}^{-1} A_{\ell i} U_m^{(i)}).$$

The generalisation of Krylov AMLS from single level to multiple levels is not as obvious as for the modified mode selection approach in the last Section. One first has to construct and solve the stationary condensed eigenvalue problem corresponding to the interface degrees on the coarsest level to obtain the initial blocks of the Krylov method for the next level problems. Next one solves the projected problems on the second level. Differently from the classical AMLS method the eigenvectors take into account the coupling between the first two levels. Again one selects a small number of eigenvectors in each substructure as initial blocks for the projected problems of the next level, and so on until finally on reaches the leaves of the sub-structuring tree.

Many questions are open:

— In classical AMLS usually on the finest levels a very small number of eigenmodes is included into the basis of the final projection space whereas the number of included substructure modes grows as the method (usually implemented bottom up) reaches coarser levels. Should one proceed in a similar fashion in Krylov AMLS?

— Are there reasonable rules how to decrease the dimensions of the initial blocks? Is this number depending only on the level of AMLS or are there further regulatory parameters?

— Is a choice based on a frequency cut-off strategy meaningful?

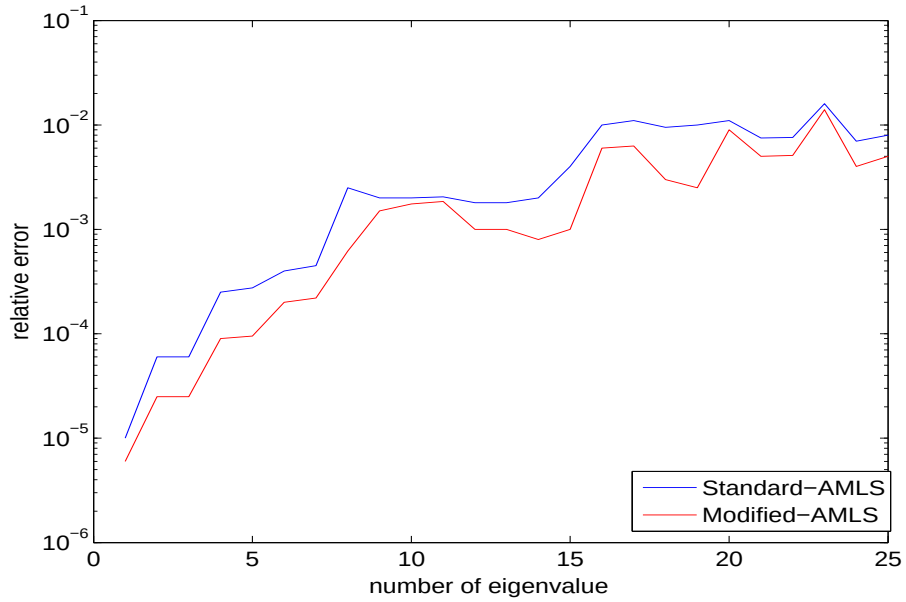

Figure 2: Comparison of standard AMLS and the modified selection of modes.

— In classical AMLS the coarsest level is reached in the final step. Hence, a very large portion the Guyan reduction to the coarsest interface is included into the the space to which the original problem is projected. Can a similar accuracy be achieved in Krylov AMLS if a much smaller number of eigenvectors on the coarsest level is included into the projection space?

These questions demonstrate that a good deal of research is still necessary to make Krylov AMLS a competitive method for large scale eigenvalue problems.

5 Numerical experiments

To evaluate the modified mode selection approach we considered a finite element discretisation of the Laplace operator on an ellipse with Neumann boundary conditions of dimension $n = 2350$. The stiffness and mass matrices where partitioned on 4 levels by bisection using METIS [12].

We determined the 25 smallest eigenvalues the maximal of which is 2.4 and we used in the standard AMLS approach a cut-off frequency of 10 (about 4 times the maximal wanted eigenvalue) yielding a reduced eigenvalue problem of dimension 243. The relative errors are shown in Figure 2 in blue.

Selecting the sub-structure eigenmodes according to the moment matching approach such that the reduced eigenvalue problem is of the same dimension we obtained the relative errors shown in Figure 2 in red. Here the maximal considered substructure eigenfrequency was 34.8 which is much larger than the cut-off frequency 10 in the

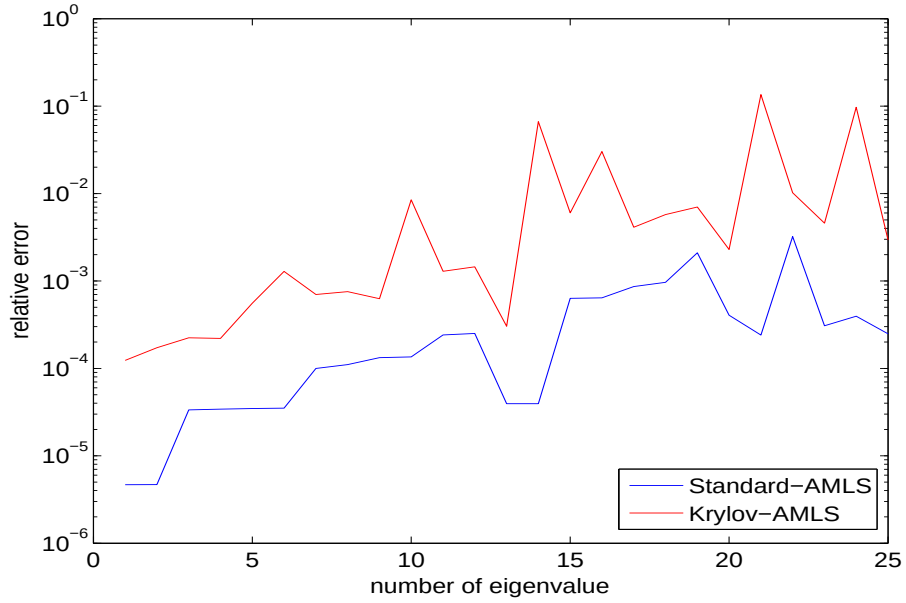

Figure 3: Comparison of standard AMLS and Krylov AMLS.

standard AMLS method. For all computed eigenvalues the relative error for the modified version of AMLS is less than the one of standard AMLS, demonstrating that the modified mode selection procedure has the potential to improve the standard AMLS method.

In our second experiment we compare the standard AMLS method with the Krylov-AMLS method. We considered the finite element model of an ore car available from MatrixMarket [14] as BCSSTK11. This is a symmetric and positive definite matrix of dimension $n = 1473$ with $nnz = 34241$ nonzero entries. METIS [12] sub-structured it into an interface of size $I_0 = 39$, and substructures of dimensions $B_1 = 700$ and $B_2 = 743$, which in turn are sub-structured into $I_1 = 45$ interface and $B_{11} = 348$ and $B_{12} = 307$ substructure degrees of freedom, and $I_2 = 42$ interface, and $B_{21} = 328$ and $B_{22} = 364$ substructure unknowns on the next level.

We reduced our problem by the Krylov-AMLS method considering 35 modes of the interface on the coarsest level, and on the following two levels we applied 8 steps of the band Krylov method to the initial block with the dominating 5 modes of the preceding level. This yielded a projected problem of dimension 275.

For the standard AMLS we tuned the cut-off frequency such that a projected problem of similar dimension occurred. The relative errors for the smallest 25 eigenvalues for standard AMLS method (in blue) and for the Krylov-AMLS approach (in red) are contained in Figure 3. For this rather small problem the standard AMLS method seems to have more favourable approximation properties than the Krylov-AMLS approach. More research (in particular on problems of higher dimension with a larger number of levels) is necessary to decide whether Krylov-AMLS nevertheless is a valuable approach.

6 Conclusions

The standard automated multi-level sub-structuring method does not consider the interaction of sub-structures and their boundaries sufficiently since the statically condensed eigenvalue problem is complimented only by eigenmodes of clamped sub-structures which do not regard the coupling. Alternatives based on a modified mode selection and on block-Krylov methods are discussed. Numerical experiments with four-level versions of AMLS demonstrate that the altered mode selection is promising. The Krylov subspace approach seems to be inferior to the standard AMLS. However, more research and more computational experience in particular on problems of higher dimension requiring a larger number of levels of sub-structuring are necessary to evaluate both approaches.

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