

# A MINMAX PRINCIPLE FOR NONLINEAR EIGENPROBLEMS DEPENDING CONTINUOUSLY ON THE EIGENPARAMETER

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**Abstract.** Variational characterizations of real eigenvalues of selfadjoint operators on a Hilbert space depending nonlinearly on an eigenparameter usually assume differentiable dependence of the operator on the eigenparameter. In this paper we generalize these results to nonlinear problems which depend only continuously on the parameter. This result is applied to a class of variational eigenvalue problems which in particular contains the vibrations of plates with attached masses.

*Dedicated to Ivo Marek on the occasion of his 75th birthday*

**1. Introduction.** In this paper we consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \tag{1.1}$$

where  $T(\lambda)$ ,  $\lambda \in J$ , is a selfadjoint and bounded operator on a Hilbert space  $\mathcal{H}$ , and  $J$  is a real open interval which may be unbounded. As in the linear case  $T(\lambda) = \lambda I - A$  a parameter  $\lambda \in J$  is called an eigenvalue of problem (1.1) if the equation (1.1) has a nontrivial solution  $x \neq 0$ .

For a wide class of linear selfadjoint operators  $A : \mathcal{H} \rightarrow \mathcal{H}$  the eigenvalues of the linear eigenvalue problem (1.1) with  $T(\lambda) := \lambda I - A$  can be characterized by three fundamental variational principles, namely by Rayleigh's principle [13], by Poincaré's minmax characterization [12], and by the maxmin principle of Courant [3], Fischer [5] and Weyl [30]. These variational characterizations of eigenvalues are known to be very powerful tools when studying selfadjoint linear operators on a Hilbert space  $\mathcal{H}$ . Bounds for eigenvalues, comparison theorems, interlacing results and monotonicity of eigenvalues can be proved easily with these characterizations, to name just a few.

These variational principles were generalized to the nonlinear eigenvalue problem (1.1) where the Rayleigh quotient  $R(x) := \langle Ax, x \rangle / \langle x, x \rangle$  of a linear problem  $Ax = \lambda x$  was replaced by the so called Rayleigh functional  $p$ , which is a homogeneous functional defined by the equation  $\langle T(p(x))x, x \rangle = 0$  for  $x \neq 0$ . Notice that in the linear case  $T(\lambda) := \lambda I - A$  this is exactly the Rayleigh quotient.

If the Rayleigh functional  $p$  is defined on the entire space  $\mathcal{H} \setminus \{0\}$  then the eigenproblem (1.1) is called overdamped. This term is motivated by the quadratic eigenvalue problem

$$T(\lambda)x = \lambda^2 Mx + \lambda Cx + Kx = 0 \tag{1.2}$$

governing the damped free vibrations of a system where  $M$ ,  $C$  and  $K$  are symmetric and positive definite corresponding to the mass, the damping and the stiffness of the system, respectively.

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Assume that the damping  $C = \alpha\tilde{C}$  depends on a parameter  $\alpha \geq 0$ . Then for  $\alpha = 0$  the system has purely imaginary eigenvalues corresponding to harmonic vibrations. Increasing  $\alpha$  the eigenvalues move into the left half plane as conjugate complex pairs corresponding to damped vibrations. Finally they reach the negative real axis as double eigenvalues where they immediately split and move into opposite directions. When eventually all eigenvalues have become real, and all eigenvalues going to the right are right of all eigenvalues moving to the left the system is called overdamped. In this case the two solutions

$$p_{\pm}(x) = (-\alpha\langle\tilde{C}x, x\rangle \pm \sqrt{\alpha^2\langle\tilde{C}x, x\rangle^2 - 4\langle Mx, x\rangle\langle Kx, x\rangle}) / (2\langle Mx, x\rangle).$$

of the quadratic equation

$$\langle T(\lambda)x, x\rangle = \lambda^2\langle Mx, x\rangle + \lambda\alpha\langle\tilde{C}x, x\rangle + \langle Kx, x\rangle = 0 \quad (1.3)$$

are real, and they satisfy  $\sup_{x \neq 0} p_-(x) < \inf_{x \neq 0} p_+(x)$ . Hence, equation (1.3) defines two Rayleigh functionals  $p_-$  and  $p_+$  corresponding to the intervals  $J_- := (-\infty, \inf_{x \neq 0} p_+(x))$  and  $J_+ := (\sup_{x \neq 0} p_-(x), 0)$ .

For overdamped systems Hadeler [6], [7] generalized Rayleigh's principle proving that the eigenvectors are orthogonal with respect to the generalized scalar product

$$[x, y] := \begin{cases} \frac{\langle (T(p(x)) - T(p(y)))x, y\rangle}{p(x) - p(y)}, & \text{if } p(x) \neq p(y) \\ \langle T'(p(x))x, y\rangle, & \text{if } p(x) = p(y) \end{cases}$$

which is symmetric, definite and homogeneous, but in general it is not bilinear.

For finite dimensional overdamped problems Duffin [4] proved Poincaré's minmax characterization for quadratic eigenproblems and Rogers [14] for the general nonlinear eigenproblem. Infinite dimensional overdamped problems were treated by Hadeler [7], [8], Langer [10], Rogers [15], Turner [19], [20], and Werner [29], who proved generalizations of both characterizations, the minmax principle of Poincaré, and the maxmin principle of Courant, Fischer and Weyl.

Barston [1] characterized a subset of the real eigenvalues of a nonoverdamped quadratic eigenproblem of finite dimension, and Werner and the author [26] studied the general nonoverdamped case and proved a minmax principle generalizing the characterization of Poincaré. The corresponding maxmin principle for nonoverdamped problems is contained in [21].

All papers mentioned above assume that the eigenproblem is differentiable with respect to  $\lambda$ . Without this assumption the minmax characterization was proved by Markus [11] and Hasanov [9] for overdamped problems. In this paper we generalize the approach in [26] to the nonoverdamped and non differentiable case. With the obtained characterizations we derive existence theorems and bounds for a class of variational eigenvalue problems generalizing corresponding results of Solov'ëv [17]. A final section of this paper applies these results to a rational eigenproblem governing free vibrations of a plate with elastically attached loads.

**2. A minmax principle.** Let  $\mathcal{H}$  be a Hilbert space with scalar product  $\langle \cdot, \cdot \rangle$ , and let  $J \subset \mathbb{R}$  be an open interval which may be unbounded. We consider the nonlinear eigenvalue problem

$$T(\lambda)x = 0 \quad (2.1)$$

where  $T(\lambda) : \mathcal{H} \rightarrow \mathcal{H}$ ,  $\lambda \in J$ , is a family of selfadjoint and bounded operators depending continuously on the parameter  $\lambda$ . As in the linear case  $T(\lambda) = \lambda I - A$  a parameter  $\lambda \in J$  is called an eigenvalue of problem (2.1) if the equation (2.1) has a nontrivial solution  $x \neq 0$ , and  $x$  is called a corresponding eigenvector. We stress the fact that we are only concerned with real eigenvalues in  $J$  although  $T(\cdot)$  may be defined on a larger subset of  $\mathbb{C}$ , and  $T(\cdot)$  may have additional eigenvalues in  $\mathbb{C} \setminus J$ .

To generalize the variational characterization of eigenvalues we need a generalization of the Rayleigh quotient. To this end we assume that

(A<sub>1</sub>) for every fixed  $x \in \mathcal{H}$ ,  $x \neq 0$  the real equation

$$f(\lambda; x) := \langle T(\lambda)x, x \rangle = 0 \quad (2.2)$$

has at most one solution  $\lambda =: p(x) \in J$ .

Then  $f(\lambda; x) = 0$  defines a functional  $p$  on some subset  $\mathcal{D}(p) \subset \mathcal{H}$  which is called the Rayleigh functional of (2.1), and which is exactly the Rayleigh quotient in case of a linear eigenproblem  $T(\lambda) = \lambda I - A$ .

Generalizing the definiteness requirement for linear pencils  $T(\lambda) = \lambda B - A$  we further assume that

(A<sub>2</sub>) for every  $x \in \mathcal{D}(p)$  and every  $\lambda \in J$  with  $\lambda \neq p(x)$  it holds that

$$(\lambda - p(x))f(\lambda, x) > 0. \quad (2.3)$$

The key to the variational principle in the nonoverdamped case is an appropriate enumeration of the eigenvalues. In general, the natural enumeration i.e. the first eigenvalue is the smallest one, followed by the second smallest one etc. is not reasonable. Instead, the number of an eigenvalue  $\lambda$  of the nonlinear problem (2.1) is inherited from the number of the eigenvalue 0 in the spectrum of the operator  $T(\lambda)$  based on the following consideration (cf. [26]).

For  $j \in \mathbb{N}$  and  $\lambda \in J$  let

$$\mu_j(\lambda) := \sup_{V \in S_j} \min_{v \in V, v \neq 0} \frac{\langle T(\lambda)v, v \rangle}{\langle v, v \rangle} \quad (2.4)$$

where  $S_j$  is the set of all  $j$  dimensional subspaces of  $\mathcal{H}$ . We assume that

(A<sub>3</sub>) If  $\mu_n(\lambda) = 0$  for some  $n \in \mathbb{N}$  and some  $\lambda \in J$ , then for  $j = 1, \dots, n$  the supremum in  $\mu_j(\lambda)$  is attained, and  $\mu_1(\lambda) \geq \mu_2(\lambda) \geq \dots \geq \mu_n(\lambda)$  are the  $n$  largest eigenvalues of the linear operator  $T(\lambda)$ . Conversely, if  $\mu = 0$  is an eigenvalue of the operator  $T(\lambda)$ , then  $\mu_n(\lambda) = 0$  for some  $n \in \mathbb{N}$ .

DEFINITION 2.1.  $\lambda \in J$  is an  $n$ th eigenvalue of  $T(\cdot)$  if  $\mu_n(\lambda) = 0$  for  $n \in \mathbb{N}$ .

Condition (A<sub>3</sub>) is satisfied for example if for every  $\lambda \in J$  there exists  $\nu(\lambda) > 0$  such that  $T(\lambda) - \nu(\lambda)I$  is a compact operator since then the classical maxmin characterization for linear selfadjoint operators applies. More general conditions are contained in [28].

Our main result is contained in the following theorem which generalizes the min-max characterization for nonoverdamped nonlinear eigenproblems in [26]. In that paper we assumed that  $T(\lambda)$  is differentiable in  $J \times \mathcal{H}$ , and it holds that

$$\left. \frac{\partial}{\partial \lambda} f(\lambda; x) \right|_{\lambda=p(x)} > 0 \quad \text{for every } x \in \mathcal{D}(p). \quad (2.5)$$

Our more general result requires some modifications of the proof given in [26].

**THEOREM 2.2.** *Assume that the conditions  $(A_1)$ ,  $(A_2)$  and  $(A_3)$  are satisfied. Then for every  $n \in \mathbb{N}$  there exists at most one  $n$ th eigenvalue, and the following characterization holds:*

$$\lambda_n = \min_{\substack{V \in \mathcal{S}_n \\ V \cap \mathcal{D}(p) \neq \emptyset}} \sup_{v \in V \cap \mathcal{D}(p)} p(v). \quad (2.6)$$

In order to prove Theorem 2.2 we need the following two Lemmata the first of which follows immediately from the implicit function theorem in the differentiable case.

**LEMMA 2.3.** *Under the conditions  $(A_1)$  and  $(A_2)$  the domain of definition  $\mathcal{D}(p)$  of the Rayleigh functional  $p$  is an open set.*

*Proof.* We assume that  $\mathcal{D}(p)$  is not an open set. Then there exists  $x \in \mathcal{D}(p)$  and a sequence  $\{x_n\} \subset \mathcal{H} \setminus \mathcal{D}(p)$  with  $x_n \rightarrow x$ , and the continuity of  $T(\lambda)$  yields

$$f(\lambda; x_n) = \langle T(\lambda)x_n, x_n \rangle \rightarrow \langle T(\lambda)x, x \rangle = f(\lambda; x)$$

for every  $\lambda \in J$ .

For  $\lambda_1 < p(x)$  it follows from  $(A_2)$  that  $f(\lambda_1; x) < 0$ , and thus it follows for sufficiently large  $n \in \mathbb{N}$  that  $f(\lambda_1; x_n) < 0$ . Analogously, we obtain  $f(\lambda_2; x_n) > 0$  for  $\lambda_2 > p(x)$  and sufficiently large  $n \in \mathbb{N}$ . Hence, for  $n$  large enough the continuous function  $f(\cdot; x_n)$  has a root in  $(\lambda_1, \lambda_2)$  contradicting  $\{x_n\} \subset \mathcal{H} \setminus \mathcal{D}(p)$ .  $\square$

**LEMMA 2.4.** *Let  $\lambda \in J$ , and assume that  $V$  is a finite dimensional subspace of  $\mathcal{H}$  such that  $V \cap \mathcal{D}(p) \neq \emptyset$ . Then it holds that*

$$\lambda \left\{ \begin{array}{l} < \\ = \\ > \end{array} \right\} \sup_{x \in V \cap \mathcal{D}(p)} p(x) \quad \Leftrightarrow \quad \min_{x \in V} \langle T(\lambda)x, x \rangle \left\{ \begin{array}{l} < \\ = \\ > \end{array} \right\} 0 \quad (2.7)$$

*Proof.* Let

$$\hat{\lambda} = \sup_{x \in V \cap \mathcal{D}(p)} p(x) \in J. \quad (2.8)$$

Then there exists a sequence  $\{x_n\} \subset \mathcal{D}(p) \cap V$  with  $p(x_n) \rightarrow \hat{\lambda}$  and  $\|x_n\| = 1$ .

Without loss of generality we assume that  $\{x_n\}$  converges to some  $\hat{x}$  (notice that the dimension of  $V$  is finite), and the continuity of  $T$  implies

$$0 = \lim_{n \rightarrow \infty} \langle T(p(x_n))x_n, x_n \rangle = \langle T(\hat{\lambda})\hat{x}, \hat{x} \rangle.$$

Hence,  $\hat{x} \in \mathcal{D}(p)$  and  $p(\hat{x}) = \hat{\lambda} = \max_{x \in \mathcal{D}(p) \cap V} p(x)$ , and it follows that

$$\min_{x \in V} \langle T(\hat{\lambda})x, x \rangle \leq \langle T(\hat{\lambda})\hat{x}, \hat{x} \rangle = 0.$$

If

$$\min_{x \in V} \langle T(\hat{\lambda})x, x \rangle = \langle T(\hat{\lambda})\hat{y}, \hat{y} \rangle < 0$$

and without loss of generality  $\langle T(\hat{\lambda})\hat{x}, \hat{y} \rangle \leq 0$ , then it follows that

$$\langle T(\hat{\lambda})(\hat{x} + t\hat{y}), \hat{x} + t\hat{y} \rangle = 2t\langle T(\hat{\lambda})\hat{x}, \hat{y} \rangle + t^2\langle T(\hat{\lambda})\hat{y}, \hat{y} \rangle < 0 \quad (2.9)$$

for every  $t > 0$ . Since  $V \cap \mathcal{D}(p)$  is an open set,  $\hat{x} + t\hat{y} \in \mathcal{D}(p)$  for sufficiently small  $t > 0$ , and (2.9) yields  $p(\hat{x} + t\hat{y}) > \hat{\lambda}$  contradicting (2.8). Hence, we have shown:

$$\lambda = \sup_{x \in V \cap \mathcal{D}(p)} p(x) \quad \Rightarrow \quad \min_{x \in V} \langle T(\lambda)x, x \rangle = 0. \quad (2.10)$$

Conversely, if

$$f(\lambda, x) = \langle T(\lambda)x, x \rangle \geq \min_{y \in V} \langle T(\lambda)y, y \rangle = 0 = \langle T(\lambda)\bar{y}, \bar{y} \rangle,$$

then  $\bar{y} \in \mathcal{D}(p)$  with  $p(\bar{y}) = \lambda$ , and condition  $(A_2)$  yields  $p(x) \leq \lambda$  for every  $x \in V \cap \mathcal{D}(p)$ . Thus, we have

$$\lambda = \sup_{x \in V \cap \mathcal{D}(p)} p(x) \quad \Leftrightarrow \quad \min_{x \in V} \langle T(\lambda)x, x \rangle = 0. \quad (2.11)$$

Let  $\min_{x \in V} \langle T(\lambda)x, x \rangle > 0$ . From  $(A_2)$  we obtain  $\lambda - p(x) > 0$  for every  $x \in V \cap \mathcal{D}(p)$ , i.e.  $\lambda \geq \sup_{x \in \mathcal{D}(p) \cap V} p(x)$ , and according to (2.11) it even holds that  $\lambda > \sup_{x \in V \cap \mathcal{D}(p)} p(x)$ .

Finally, let

$$\langle T(\lambda)\bar{y}, \bar{y} \rangle = \min_{x \in V} \langle T(\lambda)x, x \rangle < 0.$$

If  $\lambda > \sup_{x \in V \cap \mathcal{D}(p)} p(x)$ , then  $\lambda > p(\bar{x})$  for some  $\bar{x} \in V \cap \mathcal{D}(p)$ , and therefore  $\langle T(\lambda)\bar{x}, \bar{x} \rangle > 0$ . The continuous function

$$g(t) = \langle T(\lambda)((1-t)\bar{x} + t\bar{y}), (1-t)\bar{x} + t\bar{y} \rangle,$$

has opposite signs at the end points of the interval  $[0, 1]$ . Hence, there exists a root  $\bar{t} \in (0, 1)$ , and with  $w = (1-\bar{t})\bar{x} + \bar{t}\bar{y} \in V \cap \mathcal{D}(p)$  we obtain  $p(w) = \lambda$ , contradicting  $\lambda > \sup_{x \in V \cap \mathcal{D}(p)} p(x)$ , which completes the proof.  $\square$

We are now in the position to prove Theorem 2.2.

*Proof.* (of Theorem 2.2)

Let  $\lambda_n$  be an  $n$ th eigenvalue of problem (2.1). Then  $\mu_n(\lambda_n) = 0$ , and the supremum in (2.4) is attained for some  $\bar{V}_n$ , i.e.  $\min_{v \in \bar{V}_n} \langle T(\lambda_n)v, v \rangle = 0$ . For every  $V \in S_n$  we have  $\min_{v \in V} \langle T(\lambda_n)v, v \rangle \leq 0$ , and Lemma 2.4 implies

$$\sup_{x \in V \cap \mathcal{D}(p)} p(x) \geq \lambda_n = \sup_{x \in \bar{V}_n \cap \mathcal{D}(p)} p(x),$$

and therefore it holds that

$$\lambda_n = \min_{\substack{V \in S_n \\ V \cap \mathcal{D}(p) \neq \emptyset}} \sup_{v \in V \cap \mathcal{D}(p)} p(v).$$

$\square$

The following two Theorems were proved in [26]. Their proofs do not require the differentiability of  $f$ , but only that  $\mathcal{D}(p)$  is an open set and Lemma 2.4. Therefore they are also valid if assumption (2.5) is replaced by condition  $(A_2)$ .

THEOREM 2.5. *Under the conditions  $(A_1)$ ,  $(A_2)$  and  $(A_3)$  let*

$$\lambda_n := \inf_{\substack{V \in \mathcal{S}_n \\ V \cap \mathcal{D}(p) \neq \emptyset}} \sup_{v \in V \cap \mathcal{D}(p)} p(v) \in J. \quad (2.12)$$

Then  $\lambda_n$  is an  $n$ th eigenvalue of (2.1), and (2.6) holds.

THEOREM 2.6. *Assume that conditions  $(A_1)$ ,  $(A_2)$  and  $(A_3)$  are satisfied, and that  $J$  contains a first eigenvalue  $\lambda_1 = \inf_{x \in \mathcal{D}(p)} p(x)$ .*

*If  $\lambda_n \in J$  for some  $n \in \mathbb{N}$  then every  $V \in H_j$  with  $V \cap \mathcal{D}(p) \neq \emptyset$  and  $\lambda_j = \sup_{u \in V \cap \mathcal{D}(p)} p(u)$  is contained in  $\mathcal{D}(p) \cup \{0\}$ , and the characterization (2.6) can be replaced by*

$$\lambda_j = \min_{\substack{V \in H_j \\ V_1 \subset \mathcal{D}(p) \cup \{0\}}} \max_{v \in V_1} p(v) \quad j = 1, \dots, n. \quad (2.13)$$

Finally, the proof of the maxmin characterization derived in [21] does not require the differentiability of  $f$  but only Lemma 2.3, and therefore it holds that

THEOREM 2.7. *Assume that the conditions  $(A_1)$ ,  $(A_2)$  and  $(A_3)$  are satisfied. If there is an  $n$ -th eigenvalue  $\lambda_n \in J$  of problem (2.1), then*

$$\lambda_n = \max_{\substack{V \in H_{n-1} \\ V^\perp \cap \mathcal{D}(p) \neq \emptyset}} \inf_{v \in V^\perp \cap \mathcal{D}(p)} p(v), \quad (2.14)$$

and the maximum is attained by  $W := \text{span}\{u_1, \dots, u_{n-1}\}$  where  $u_j$  denotes an eigenvector corresponding to the  $j$ -largest eigenvalue  $\mu_j(\lambda_n)$  of  $T(\lambda_n)$ .

**3. A variational eigenvalue problem.** We consider the nonlinear eigenvalue problem in variational form

Find  $\lambda \in \mathbb{R}$  and  $u \in \mathcal{H}$ ,  $u \neq 0$ , such that

$$a(u, v, \lambda) = \lambda b(u, v, \lambda) \quad \text{for every } v \in \mathcal{H}. \quad (3.1)$$

We assume that the following conditions are satisfied:

- (i)  $a : \mathcal{H} \times \mathcal{H} \times J \rightarrow \mathbb{R}$  is an  $\mathcal{H}$ -elliptic, continuous and symmetric bilinear form, i.e. there exist positive functions  $\alpha_1, \alpha_2 : J \rightarrow \mathbb{R}^+$  such that for  $\lambda \in J$  it holds that

$$\alpha_1(\lambda) \|u\|^2 \leq a(u, u, \lambda), \quad \text{and } |a(u, v, \lambda)| \leq \alpha_2(\lambda) \|u\| \cdot \|v\| \quad \text{for every } u, v \in \mathcal{H}.$$

- (ii)  $b : \mathcal{H} \times \mathcal{H} \times J \rightarrow \mathbb{R}$  is a symmetric, completely continuous and positive definite bilinear form on  $\mathcal{H}$ , i.e. there exists  $\beta_2 : J \rightarrow \mathbb{R}$  such that

$$\begin{aligned} 0 < b(u, v, \lambda) & \quad \text{for every } u \in \mathcal{H} \setminus \{0\} \\ \text{and } |b(u, v)| & \leq \beta_2(\lambda) \|u\| \cdot \|v\| \quad \text{for every } u, v \in \mathcal{H}, \end{aligned}$$

and if  $\{u_n\}, \{v_n\} \subset \mathcal{H}$  are weakly convergent sequences such that  $u_n \rightharpoonup u$ ,  $v_n \rightharpoonup v$  then  $b(u_n, v_n, \lambda) \rightarrow b(u, v, \lambda)$ .

- (iii) With respect to the eigenparameter  $\lambda$ , we assume that  $a$  and  $b$  are uniformly continuous, i.e.

$$\lim_{\mu \rightarrow \lambda} \alpha(\lambda, \mu) = 0 \quad \text{and} \quad \lim_{\mu \rightarrow \lambda} \beta(\lambda, \mu) = 0$$

where

$$|a(u, v, \lambda) - a(u, v, \mu)| \leq \alpha(\lambda, \mu) \|u\| \|v\|, \quad |b(u, v, \lambda) - b(u, v, \mu)| \leq \beta(\lambda, \mu) \|u\| \|v\|.$$

It is well known (cf. Weinberger [27]) that for fixed  $\lambda \in J$  there exist a bounded operator  $A(\lambda) : \mathcal{H} \rightarrow \mathcal{H}$  and a completely continuous operator  $B(\lambda) : \mathcal{H} \rightarrow \mathcal{H}$  such that

$$a(u, v, \lambda) = \langle A(\lambda)u, v \rangle \quad \text{and} \quad b(u, v, \lambda) = \langle B(\lambda)u, v \rangle \quad \text{for every } u, v \in \mathcal{H}.$$

Hence, the variational eigenvalue problem (3.1) is equivalent to problem (2.1) with

$$T(\lambda) = \lambda B(\lambda) - A(\lambda), \quad (3.2)$$

and it follows from (iii) that  $A$  and  $B$  (and therefore  $T$ ) depend continuously on  $\lambda$ . From the maxmin characterization of Poincaré for linear eigenproblems it follows that condition  $(A_3)$  is satisfied.

For fixed  $u \in \mathcal{H}$

$$f(\lambda; u) = \lambda b(u, u, \lambda) - a(u, u, \lambda), \quad (3.3)$$

and the general conditions in Section 2 are fulfilled, if for fixed  $u \in \mathcal{H}$  the real equation  $f(\lambda; u) = 0$  has at most one solution  $p(u)$  in  $J$ , and if condition (2.3) holds. This is, for example, the case if

(iv) for fixed  $u \in \mathcal{H} \setminus \{0\}$  the Rayleigh quotient

$$R(\lambda; u) = \frac{a(u, u, \lambda)}{b(u, u, \lambda)} \quad (3.4)$$

of the linear eigenproblem  $a(u, v, \lambda) = \mu b(u, v, \lambda)$  is monotonically not increasing with respect to  $\lambda \in J$ .

Under the conditions (i), (ii), (iii), and (iv) the linear eigenvalue problem: find  $u \neq 0$  and  $\gamma$  such that

$$a(u, v, \mu) = \gamma(\mu) b(u, v, \mu) \quad \text{for every } v \in \mathcal{H}$$

has an infinite number of eigenvalues  $0 < \gamma_1(\mu) \leq \gamma_2(\mu) \leq \dots$  which depend continuously on  $\mu$  and which are monotonically decreasing with respect to  $\mu$ . Obviously  $\mu$  is an eigenvalue of the nonlinear eigenvalue problem (3.1) if and only if  $\mu = \gamma_j(\mu)$  for some  $j$ . This was the basis of existence results for (3.1) proved by Solov'ev [17, 18]. Our results for the nonlinear eigenproblem (3.1) are based on the following lemma:

LEMMA 3.1. *Assume that (i), (ii), (iii) and (iv) are satisfied, and let  $u \in \mathcal{H} \setminus \{0\}$  with  $R(\kappa, u) \in J$  for some  $\kappa \in J$ . Then  $u \in \mathcal{D}(p)$  and the following inequalities hold*

$$\min(\kappa, R(\kappa, u)) \leq p(u) \leq \max(\kappa, R(\kappa, u)). \quad (3.5)$$

*Proof.* Condition (iv) yields

$$\begin{aligned} \frac{f(R(\kappa, u), u)}{b(u, u, R(\kappa, u))} &= R(\kappa, u) - \frac{a(u, u, R(\kappa, u))}{b(u, u, R(\kappa, u))} \\ &= R(\kappa, u) - R(R(\kappa, u), u) \begin{cases} \leq \\ \geq \end{cases} \left. \vphantom{\frac{f(R(\kappa, u), u)}{b(u, u, R(\kappa, u))}} \right\} 0 \text{ for } \begin{cases} R(\kappa, u) \leq \kappa \\ R(\kappa, u) \geq \kappa \end{cases} \end{aligned}$$

and similarly

$$\frac{f(\kappa, u)}{b(u, u, \kappa)} = \kappa - R(\kappa, u) \left\{ \begin{array}{l} \leq \\ \geq \end{array} \right\} 0 \text{ for } \left\{ \begin{array}{l} R(\kappa, u) \geq \kappa \\ R(\kappa, u) \leq \kappa \end{array} \right\}.$$

In either case,  $R(\kappa, u) \geq \kappa$  and  $R(\kappa, u) \leq \kappa$ ,  $f(\cdot, u)$  changes its sign between  $R(\kappa, u)$  and  $\kappa$ , and therefore  $u \in \mathcal{D}(p)$  and the inclusion (3.5) holds.  $\square$

We first consider the case that (3.1) has a first eigenvalue in  $J$ .

**THEOREM 3.2.** *Assume that (i), (ii), (iii) and (iv) hold, and let  $\kappa \in J$ . Suppose that there exists  $\eta \in J$  such that*

$$\eta - \min_{u \in \mathcal{H} \setminus \{0\}} R(\eta, u) \leq 0, \quad (3.6)$$

and that the linear eigenvalue problem:

find  $\mu \in \mathbb{R}$  and  $u \in \mathcal{H} \setminus \{0\}$  with

$$a(u, v, \kappa) = \mu b(u, v, \kappa) \quad \text{for every } v \in \mathcal{H} \quad (3.7)$$

has  $r$  eigenvalues  $\mu_1 \leq \mu_2 \leq \dots \leq \mu_r$  in  $J$ .

Then the nonlinear eigenvalue problem (3.1) has  $r$  eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$  in  $J$ , and the following inclusion holds

$$\min(\mu_j, \kappa) \leq \lambda_j \leq \max(\mu_j, \kappa), \quad j = 1, \dots, r. \quad (3.8)$$

*Proof.* We first proof the existence of  $r$  eigenvalues of (3.1).

From (3.6) it follows that

$$\inf_{u \in \mathcal{D}(p)} p(u) \geq \eta.$$

Hence, by Theorem 2.6 we only have to show that there exist a subspace  $W$  of dimension  $r$  with  $W \cap \mathcal{D}(p) \neq \emptyset$  and  $\sup_{u \in W \cap \mathcal{D}(p)} p(u) \in J$ .

Let  $W$  be the invariant subspace of problem (3.7) which is spanned by the eigenvectors corresponding to  $\mu_1, \dots, \mu_r$ . Then it holds that

$$\mu_r = \min_{\dim V=r} \max_{v \in V, v \neq 0} R(\kappa, v) = \max_{v \in W, v \neq 0} R(\kappa, v) \in J.$$

Hence,  $\eta \leq R(\kappa, v) \leq \mu_r$  for every  $v \in W \setminus \{0\}$ , and it follows from Lemma 3.1 that  $W \setminus \{0\} \subset \mathcal{D}(p)$ , and

$$p(v) \leq \max(\kappa, R(\kappa, v)) \leq \max(\kappa, \mu_r).$$

By Theorem 2.6 problem (3.1) has at least  $r$  eigenvalues  $\lambda_1 \leq \dots \leq \lambda_r$  in  $J$ .

We now prove inequality (3.8). For  $j \in \{1, \dots, r\}$  let  $Z_j$  be the  $j$  dimensional subspace with

$$\mu_j = \min_{\dim V=j} \max_{v \in V, v \neq 0} R(\kappa, v) = \max_{v \in Z, z \neq 0} R(\kappa, v).$$

Then it holds that  $Z \subset W \subset \mathcal{D}(p) \cup \{0\}$ , and from

$$p(v) \leq \max(\kappa, R(\kappa, v)) \quad \text{for every } v \in Z, z \neq 0$$



we obtain

$$\begin{aligned}\lambda_j &= \min_{\substack{\dim V = j \\ V \subset \mathcal{D}(p) \cup \{0\}}} \max_{v \in V, v \neq 0} p(v) \leq \max_{v \in Z, v \neq 0} p(v) \\ &\leq \max(\kappa, \max_{v \in Z, v \neq 0} R(\kappa, v)) = \max(\kappa, \mu_j),\end{aligned}$$

which proves the upper bound of  $\lambda_j$  in (3.8).

Let  $Y \subset \mathcal{D}(p) \cup \{0\}$ ,  $\dim Y = j$  with

$$\lambda_j = \min_{\substack{\dim V = j \\ V \subset \mathcal{D}(p) \cup \{0\}}} \max_{v \in V, v \neq 0} p(v) = \max_{v \in Y, v \neq 0} p(v).$$

For  $u \in Y$ ,  $u \neq 0$  with  $p(u) \leq \kappa$  we obtain

$$p(u) = \frac{a(u, u, p(u))}{b(u, u, p(u))} \geq \frac{a(u, u, \kappa)}{b(u, u, \kappa)} = R(\kappa, u).$$

Hence, if  $p(u) \leq \kappa$  for every  $u \in Y$ ,  $u \neq 0$ , then

$$\lambda_j = \max_{u \in Y, u \neq 0} p(u) \geq \max_{u \in Y, u \neq 0} R(\kappa, u) \geq \min_{\dim V = j} \max_{u \in V, u \neq 0} R(\kappa, u) = \mu_j,$$

and if there exists  $u \in Y$ ,  $u \neq 0$  with  $p(u) > \kappa$ , then it follows  $\lambda_j = \max_{v \in Y, v \neq 0} p(v) \geq p(u) > \kappa$ . In either case we have

$$\lambda_j \geq \min(\kappa, \mu_j),$$

which completes the proof.  $\square$

Next we consider the case that the interval  $J$  does not contain a first eigenvalue.

We first prove the following result

**THEOREM 3.3.** *Assume that the conditions (i), (ii), (iii), and (iv) are satisfied, and let  $\kappa \in J$ .*

*Suppose that  $\kappa \in J$  and  $J$  contains an  $m$ th eigenvalue  $\mu_m$  of the linear eigenproblem (3.7), then the nonlinear eigenvalue problem (3.1) has an  $m$ th eigenvalue  $\lambda_m \in J$ , and it holds that*

$$\min(\mu_m, \kappa) \leq \lambda_m \leq \max(\mu_m, \kappa). \quad (3.9)$$

*Proof.* We show that

( $\alpha$ ) there exists a subspace  $W$ ,  $\dim W = m$  with  $W \cap \mathcal{D}(p) \neq \emptyset$  and

$$\sup_{u \in W \cap \mathcal{D}(p)} p(u) \leq \max(\kappa, \mu_m) \quad (3.10)$$

( $\beta$ )

$$\sup_{u \in V \cap \mathcal{D}(p)} p(u) \geq \min(\kappa, \mu_m) \quad \text{for every } V \text{ with } \dim V = m \text{ and } V \cap \mathcal{D}(p) \neq \emptyset.$$

Then it holds that

$$\lambda_m := \inf_{\substack{\dim V = m \\ V \cap \mathcal{D}(p) \neq \emptyset}} \sup_{u \in V \cap \mathcal{D}(p)} p(u) \in J,$$

and the inclusion (3.9) follows.

( $\alpha$ ): Let  $\dim W = m$ , and let  $w \in W$  with

$$\mu_m = \max_{u \in W, u \neq 0} R(\kappa, u) = R(\kappa, w).$$

Then Lemma 3.1 yields  $w \in \mathcal{D}(p)$ , and  $W \cap \mathcal{D}(p) \neq \emptyset$ .

$R(\kappa, u) \leq \mu_m$  for every  $u \in W$ ,  $u \neq 0$  implies

$$\frac{a(u, u, \kappa)}{b(u, u, \kappa)} \leq \mu_m \quad \text{for every } u \in W, u \neq 0.$$

If  $\kappa \geq \mu_m$ , it follows immediately  $f(\kappa, u) \geq 0$ , and if  $\mu_m \geq \kappa$ , then we obtain  $f(\mu_m, u) \geq 0$  from the monotonicity of  $R(\lambda, u)$ . In either case it holds that  $f(\sigma, u) \geq 0$  for  $\sigma = \max(\kappa, \mu_m)$  for every  $u \in W$ ,  $u \neq 0$  and condition (ii) yields inequality (3.10).

( $\beta$ ) is proved by contradiction: Assume that there exists  $V$ ,  $\dim V = m$  with  $V \cap \mathcal{D}(p) \neq \emptyset$  and

$$\sup_{u \in V \cap \mathcal{D}(p)} p(u) < \min(\kappa, \mu_m). \quad (3.11)$$

Let  $u_V \in V$ ,  $u_V \neq 0$  with

$$\rho := R(\kappa, u_V) = \max_{u \in V, u \neq 0} R(\kappa, u).$$

Then  $\rho \notin J$ , since otherwise by Lemma 3.1 it would follow that  $u_V \in \mathcal{D}(p)$  and  $p(u_V) \geq \min(\kappa, R(\kappa, u_V))$ , i.e.

$$\sup_{u \in V \cap \mathcal{D}(p)} p(u) \geq p(u_V) \geq \min(\kappa, R(\kappa, u_V)) \geq \min(\kappa, \mu_m).$$

Let  $\sigma := \min(\kappa, \mu_m)$ . Then  $\sigma \leq \rho$  implies

$$\frac{f(\sigma, u_V)}{b(u_V, u_V, \sigma)} = \sigma - \frac{a(u_V, u_V, \sigma)}{b(u_V, u_V, \sigma)} \leq \sigma - \rho \leq 0,$$

i.e.  $f(\sigma, u_V) \leq 0$ . For fixed  $u \in V \cap \mathcal{D}(p)$  let  $w(t) := tu + (1-t)u_V$  and  $\phi(t) = f(\sigma, w(t))$ . Then  $\phi$  is continuous on  $[0, 1]$ , and it holds that

$$f(\sigma, u_V) = \phi(0) \leq 0 \leq f(\sigma, u) = \phi(1).$$

Hence, there exists  $\hat{t} \in [0, 1]$  with  $f(\sigma, w(\hat{t})) = 0$ , i.e.  $w(\hat{t}) \in V \cap \mathcal{D}(p)$  and  $p(w(\hat{t})) = \min(\kappa, \mu_m)$  contradicting (3.11).  $\square$

From Theorem 3.2 we immediately obtain

**THEOREM 3.4.** *Assume that (i), (ii), (iii), and (iv) are satisfied. Let  $\kappa_1, \kappa_2 \in J$  with  $\kappa_1 < \kappa_2$ , and suppose that for  $\kappa = \kappa_j$ ,  $j = 1, 2$  the linear eigenvalue problem (3.7) has exactly  $m_j$  eigenvalues which do not exceed  $\kappa_j$*

*Then the nonlinear eigenvalue problem (3.1) has at least  $m_2 - m_1$  eigenvalues in  $J$ .*

*In  $(\kappa_1, \kappa_2]$  there are exactly  $m_2 - m_1$  eigenvalue  $\lambda_{m_1+1} \leq \dots \leq \lambda_{m_2}$ , where  $\lambda_j$  is a  $j$ th eigenvalue, and for every parameter  $\kappa \in [\kappa_1, \kappa_2]$  it holds that*

$$\min(\kappa, \mu_j(\kappa)) \leq \lambda_j \leq \max(\kappa, \mu_j(\kappa)), \quad \text{für } j = m_1 + 1, \dots, m_2. \quad (3.12)$$

*Here  $\mu_j(\kappa)$  denotes the  $j$  smallest eigenvalue of (3.7) corresponding to  $\kappa$ .*

*Proof.* Follows immediately from Theorems 3.3 und 2.6.  $\square$

**4. Vibration of plates with masses.** The vertical deflection  $w(x, t)$  of a thin isotropic clamped plate with elastically attached loads is governed by the equations [16, 24]

$$Lw(x, t) + \rho h \frac{\partial^2}{\partial t^2} w(x, t) + \sum_{j=1}^q m_j \frac{d^2}{dt^2} \xi_j(t) \delta_{x_j} w = 0, x \in \Omega, t > 0 \quad (4.1)$$

$$w(x, t) = \partial_n w(x, t) = 0, x \in \partial\Omega, t > 0 \quad (4.2)$$

$$m_j \frac{d^2}{dt^2} \xi_j + k_j (\xi_j(t) - w(x_j, t)) = 0, t > 0, j = 1, \dots, q. \quad (4.3)$$

Here  $\Omega \subset \mathbb{R}^2$  is a bounded Lipschitz domain which is occupied by the plate,  $\rho$  is the mass per volume density, and  $h$  the thickness of the plate.  $L = D\Delta^2$  is the plate operator,  $D = Eh^3/12(1 - \nu^2)$  is the flexural rigidity of the plate,  $E$  is the Young modulus and  $\nu$  is the Poisson number. For  $j = 1, \dots, q$  at  $x_j \in \Omega$  a load  $m_j$  is joined elastically to the plate with stiffness coefficient  $k_j$ , and  $\xi_j$  denotes the displacement of the mass  $m_j$ .

Using the ansatz  $w(x, t) = u(x)e^{i\omega t}$  and  $\xi_j(t) = c_j e^{i\omega t}$  characterizing the eigenmodes and eigenfrequencies of the vibrating plate, and eliminating  $c_j$  we obtain the rational eigenproblem

$$Lu(x) = \lambda \rho h u(x) + \sum_{j=1}^q \frac{\lambda \sigma_j}{\sigma_j - \lambda} m_j \delta_{x_j} u, x \in \Omega \quad (4.4)$$

$$u(x) = \partial_n u(x) = 0, x \in \partial\Omega \quad (4.5)$$

where  $\lambda = \omega^2$  and  $\sigma_j = k_j/m_j$ .

Let  $\mathcal{H} = H_0^2(\Omega)$  denote the Sobolev space with the usual scalar product. Multiplying (4.4) by  $v \in \mathcal{H}$  and integrating by parts one obtains the variational form of problem (4.4), (4.5)

$$A(u, v) = \lambda \left( B(u, v) + \sum_{j=1}^p \frac{\sigma_j}{\sigma_j - \lambda} C_j(u, v) \right) \quad (4.6)$$

where

$$A(u, v) = \int_{\Omega} D\Delta u \Delta v \, dx, \quad (4.7)$$

$$B(u, v) = \int_{\Omega} \rho h u v \, dx, \quad (4.8)$$

$$C_j(u, v) = m_j u(x_j) v(x_j). \quad (4.9)$$

For  $J = (0, \min_j \sigma_j)$  the conditions of Theorem 3.2 are obviously satisfied with

$$a(u, v, \lambda) = A(u, v) \quad \text{and} \quad b(u, v, \lambda) = B(u, v) + \sum_{j=1}^p \frac{\sigma_j}{\sigma_j - \lambda} C_j(u, v).$$

Hence, if for some  $\kappa \leq \min_j \sigma_j$  the linear eigenproblem: find  $u \in H_0^2(\Omega)$  and  $\mu \in \mathbb{R}$  such that

$$A(u, v) = \mu \left( B(u, v) + \sum_{j=1}^p \frac{\sigma_j}{\sigma_j - \kappa} C_j(u, v) \right) \quad \text{for every } v \in H_0^2(\Omega) \quad (4.10)$$

has  $r$  eigenvalues  $0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_r < \min_j \sigma_j$ , then the nonlinear eigenproblem (4.4), (4.5) has  $r$  eigenvalues  $\lambda_j$ , and

$$\min(\mu_j, \kappa) \leq \lambda_j \leq \max(\mu_j, \kappa).$$

Since the Rayleigh quotient of problem (4.10) is less than the one of the pure plate problem without masses:

$$\text{find } u \neq 0 \text{ and } \lambda \text{ such that } A(u, v) = \lambda B(u, v) \quad \text{for every } v \in H_0^2(\Omega), \quad (4.11)$$

it follows immediately that the nonlinear eigenproblem (4.4), (4.5) has at least as many eigenvalues in  $J$  as (4.11).

If  $J = (\xi, \eta) \subset \mathbb{R}^+$  is an interval which does not contain a pole  $\sigma_j$  then the conditions of Theorem 3.4 are satisfied with

$$a(u, v, \lambda) = A(u, v) + \sum_{\sigma_j \leq \xi} \frac{\lambda \sigma_j}{\lambda - \sigma_j} C_j(u, v), \quad (4.12)$$

$$b(u, v, \lambda) = B(u, v) + \sum_{\sigma_j \geq \eta} \frac{\sigma_j}{\sigma_j - \lambda} C_j(u, v), \quad (4.13)$$

and the number of eigenvalues of (4.4), (4.5) and enclosures of the eigenvalues can be obtained from linear eigenproblems. In particular, the interval  $J = (\max_j \sigma_j, \infty)$  contains an infinite number of eigenvalues.

In a similar fashion one gets existence results and minmax characterizations of eigenvalues of rational eigenvalue problems governing free vibrations of fluid-solid structures as considered in Conca, Planchard, Vanninathan [2, 21, 22, 25].

**NUMERICAL EXAMPLE** Consider the clamped plate occupying the domain  $\Omega = (0, 4) \times (0, 3)$  with constant coefficients  $\rho = d = 1$ . We assume that 6 masses are attached to the plate at  $x_1 = (1, 1)$ ,  $x_2 = (2, 1)$ ,  $x_3 = (3, 1)$ ,  $x_4 = (1, 2)$ ,  $x_5 = (2, 2)$  and  $x_6 = (3, 2)$ , where  $\sigma_1 = \sigma_2 = \sigma_3 = 1000$ ,  $\sigma_4 = \sigma_6 = 2000$ , and  $\sigma_5 = 3000$ , and  $m_1 = m_2 = m_3 = 1$ ,  $m_4 = m_6 = 1/2$  and  $m_5 = 1/3$ .

We discretized the eigenproblem by Bogner-Fox-Schmit elements on a quadratic mesh with stepsize  $h = 0.05$  which yielded a matrix eigenvalue problem

$$Kx = \lambda Mx + \frac{1000\lambda}{1000 - \lambda} C_1 x + \frac{1000\lambda}{2000 - \lambda} C_2 x + \frac{1000\lambda}{3000 - \lambda} C_3 x \quad (4.14)$$

of dimension 18644. Here  $K$  and  $M$  are the stiffness and the mass matrix of the plate without masses, and  $C_j$  for  $j = 1, 2, 3$  is a diagonal matrix of rank 3, 2 and 1, respectively, corresponding to the loads  $\{m_1, m_2, m_3\}$ ,  $\{m_4, m_6\}$  and  $m_5$ .

For  $\kappa := \sigma_1 - \varepsilon$ ,  $\varepsilon > 0$  small enough the linear eigenvalue problem

$$Kx = \lambda \left( Mx + \frac{1000}{1000 - \kappa} C_1 x + \frac{1000}{2000 - \kappa} C_2 x + \frac{1000}{3000 - \kappa} C_3 \right) x \quad (4.15)$$

has 24 eigenvalues which are less than  $\sigma_1 - \varepsilon$ , and by Theorem 3.2 problem (4.14) has 24 eigenvalues smaller than the smallest pole  $\sigma_1$  enumerated  $\lambda_1, \dots, \lambda_{24}$ . By the min-max characterization the eigenvalues of (4.14) are upper bounds of the corresponding eigenvalues of (4.10), and therefore (4.10) has at least 24 eigenvalues not exceeding  $\sigma_1$ .

In a similar manner one gets from Theorem 3.4 with (3.7) (where  $a$  and  $b$  are discrete versions of (4.12) and (4.13)) that the interval  $J_2 = (\sigma_1, \sigma_2)$  contains 8 eigenvalues of (4.14) enumerated  $\lambda_{22}, \dots, \lambda_{29}$ , and the interval  $J_3$  contains 9 eigenvalues

enumerated  $\hat{\lambda}_{29}, \dots, \hat{\lambda}_{36}$ . Finally, there are 18609 eigenvalues greater than  $\sigma_3$  the smallest of which being a 36th one. This example demonstrates that the enumeration of eigenvalues in  $(A_3)$  corresponds to the natural ordering only if  $\inf_{x \in \mathcal{D}(p)} p(x)$  is contained in the corresponding interval, and that there may exist eigenvalues which carry the same number but are different from each other ( $\lambda_{22} \neq \hat{\lambda}_{22}$ , e.g.). Notice that eigenvalues in  $(0, \sigma_1)$ ,  $(\sigma_1, \sigma_2)$ ,  $(\sigma_2, \sigma_3)$ , and  $(\sigma_3, \infty)$  can be determined safely by the nonlinear Arnoldi method [23, 24].

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