# Automated Multi-Level Substructuring for a Fluid-Solid Vibration Problem

Markus Stammberger and Heinrich Voss

**Abstract** The Automated Multi-Level Substructuring (AMLS) method has been developed to reduce the computational demands of frequency response analysis and has recently been proposed as an alternative to iterative projection methods like Lanczos or Jacobi–Davidson for computing a large number of eigenvalues for huge symmetric eigenvalue problems. Based on Schur complements and modal approximations of submatrices on several levels AMLS constructs a projected eigenproblem which yields good approximations of eigenvalues at the lower end of the spectrum. In this paper we discuss a structure preserving AMLS variant for nonsymmetric eigenproblems governing free vibrations of fluid–solid structures.

## 1 Introduction

Over the last few years, a new method for huge linear eigenvalue problems

$$Kx = \lambda Mx \tag{1}$$

where $K \in \mathbb{C}^{n\times n}$ and $M \in \mathbb{C}^{n\times n}$ are Hermitian and positive definite, known as *Automated Multi–Level Substructuring (AMLS)*, has been developed by Bennighof and co-authors, and has been applied to frequency response analysis of complex structures [1, 2]. Here the large finite element model is recursively divided into very many substructures on several levels based on the sparsity structure of the system matrices. Assuming that the interior degrees of freedom of substructures depend

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quasistatically on the interface degrees of freedom, and modelling the deviation from quasistatic dependence in terms of a small number of selected substructure eigenmodes the size of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest. Recent studies ([7, 11], e.g.) in vibro-acoustic analysis of passenger car bodies, where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed, have shown that for this type of problems AMLS is considerably faster than Lanczos type approaches.

On each level of the hierarchical substructuring AMLS consists of two steps. First for every substructure of the current level a congruence transformation is applied to the matrix pencil to decouple in the stiffness matrix the substructure from the degrees of freedom of higher levels. Secondly, the dimension of the problem is reduced by modal truncation of the corresponding diagonal blocks discarding eigenmodes according to eigenfrequencies which exceed a predetermined cut-off frequency. Hence, AMLS is nothing else but a projection method where the large problem under consideration is projected to a search space spanned by a smaller number of eigenmodes of clamped substructures on several levels.

Nonsymmetric eigenproblems governing free vibrations of fluid-solid structures are covered in the following way [7, 11]: one first solves the symmetric eigenproblems governing free vibrations of the fluid and the structure independently, and the original problem is then projected to the space spanned by these eigenmodes. So, the coupling is not considered when constructing the search space, but only in the projected problem. In this paper we propose an AMLS variant which incorporates the coupling already into the reduction process.

The paper is organized as follows. Section 2 summarizes the automated multilevel substructuring method for linear eigenvalue problems. Section 3 introduces the nonsymmetric eigenvalue problem governing free vibrations of a fluid-solid structure and presents the usual approach for solving it, and Section 4 proposes the structure preserving variant of AMLS. The paper closes with a numerical example demonstrating the improvement by the new approach.

## 2 Automated Multi-Level Substructuring

In this section we summarize the *Automated Multi-Level Substructuring* (AMLS) method for the linear eigenvalue problem

$$Kx = \lambda Mx \tag{2}$$

which was developed by Bennighof and co-workers [1, 2] over the last few years, who applied it to solve frequency response problems involving large and complex models. Here, $K$ is the stiffness matrix and $M$ the mass matrix of a finite element model of a structure. Both matrices are assumed to be large and sparse, and are symmetric and positive definite.

We first consider the component mode synthesis method (CMS method) which is the essential building block of the AMLS method. Assume that the graph of the matrix $|K|+|M|$ is partitioned into substructures. We distinguish only between local (i.e. interior) and interface degrees of freedom. Then $K$ and $M$ (after reordering) have the following form:

$$K = \begin{pmatrix} K_{\ell\ell} & K_{\ell i} \\ K_{i\ell} & K_{ii} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} M_{\ell\ell} & M_{\ell i} \\ M_{i\ell} & M_{ii} \end{pmatrix} \tag{3}$$

where $K_{\ell\ell}$ and $M_{\ell\ell}$ are block diagonal.

Annihilating $K_{\ell i}$ by block Gaussian elimination and transforming the local coordinates to modal degrees of freedom one obtains the equivalent pencil

$$(P^T K P, P^T M P) = \left( \begin{pmatrix} \Phi & 0 \\ 0 & \tilde{K}_{ii} \end{pmatrix}, \begin{pmatrix} I & \tilde{M}_{\ell i} \\ \tilde{M}_{i\ell} & \tilde{M}_{ii} \end{pmatrix} \right) \quad \text{with } P = \begin{pmatrix} \Phi & -K_{\ell\ell}^{-1} K_{\ell i} \\ 0 & I \end{pmatrix}. \tag{4}$$

Here $\Omega$ is a diagonal matrix containing the substructure eigenvalues, i.e. $K_{\ell\ell}\Phi = M_{\ell\ell}\Phi\Omega$, $\Phi^T M_{\ell\ell}\Phi = I$, and $\Phi$ contains in its columns the corresponding eigenvectors. Notice that $K_{\ell\ell}$ and $M_{\ell\ell}$ are block diagonal, and therefore it is quite inexpensive to eliminate $K_{\ell i}$ and to solve the interior eigenproblems.

In structural dynamics (4) is called Craig–Bampton form of the eigenvalue problem (2) corresponding to the partitioning (3).

Selecting some eigenmodes of problem (4) (usually the ones associated with eigenvalues below a cut off threshold. However, in a recent paper Lia, Bai and Gao [9] suggested a different choice based on a moment–matching analysis), and dropping the rows and columns in (4) corresponding to the other modes one arrives at the component mode synthesis method (CMS) introduced by Hurty [6] and Craig and Bampton [4]. Hence, if the diagonal matrix $\Omega$ contains on its diagonal the eigenvalues to keep and $\Phi$ the corresponding eigenvectors, then the CMS approximations to the eigenpairs of (2) are obtained from the reduced eigenvalue problem

$$\begin{pmatrix} \Omega & 0 \\ 0 & \tilde{K}_{ii} \end{pmatrix} y = \lambda \begin{pmatrix} I & \tilde{M}_{\ell i} \\ \tilde{M}_{i\ell} & \tilde{M}_{ii} \end{pmatrix} y \tag{5}$$

AMLS generalizes CMS in the following way. Again the graph of $|K|+|M|$ is partitioned into a small number of subgraphs, but more generally than in CMS these subgraphs in turn are substructured on a number $p$ of levels. This induces the following partitioning of the index set $I = \{1,\ldots,n\}$ of degrees of freedom. $I_1$ is the set of indices corresponding to interface degrees of freedom on the coarsest level, and for $j = 2,\ldots,p$ define $I_j$ to be the set of indices of interface degrees of freedom on the $j$-th level which are not contained in $I_{j-1}$. Finally, let $I_{p+1}$ be the set of interior degrees of freedom on the finest level.

With these notations the first step of AMLS is CMS with cut-off frequency $\gamma$ applied to the finest substructuring. After $j$ steps, $1 \le j \le p-1$, one derives a reduced pencil

$$\left( \begin{pmatrix} \Omega_p & O & O \\ O & K_{\ell\ell}^{(j)} & K_{\ell i}^{(j)} \\ O & K_{i\ell}^{(j)} & K_{ii}^{(j)} \end{pmatrix}, \begin{pmatrix} M_{pp}^{(j)} & M_{p\ell}^{(j)} & M_{pi}^{(j)} \\ M_{\ell p}^{(j)} & M_{\ell\ell}^{(j)} & M_{\ell i}^{(j)} \\ M_{ip}^{(j)} & M_{i\ell}^{(j)} & M_{ii}^{(j)} \end{pmatrix} \right). \tag{6}$$

where $p$ denotes the degrees of freedom obtained in the spectral reduction in the previous steps, $\ell$ collects the indices in $I_{p+1-j}$, and $i$ corresponds to the index set $\cup_{k=1}^{p-j} I_k$ of interface degrees of freedom on levels which are not yet treated. Applying the CMS method to the south–east $2 \times 2$ blocks of the matrices, i.e. annihilating the off–diagonal block $K_{\ell i}^{(j)}$ by block Gaussian elimination, and reducing the set of $\ell$–indices by spectral truncation with cut-off frequency $\gamma$ one arrives at the next level. After $p$ CMS steps and a final spectral truncation of the lower–right blocks one obtains the reduction of problem (2) by AMLS.

## 3 Fluid-solid vibrations

Vibrations of fluid-solid structures are governed by the linear eigenvalue problem

$$\begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix} \begin{pmatrix} u_s \\ p_f \end{pmatrix} = \lambda \begin{pmatrix} M_s & 0 \\ -\rho C^T & M_f \end{pmatrix} \begin{pmatrix} u_s \\ p_f \end{pmatrix} \tag{7}$$

where $K_s$ and $K_f$ are the stiffness matrices, and $M_s$ and $M_f$ are the mass matrices of the structure and the fluid, respectively. $u_s$ is the structure displacement vector, $p_f$ the fluid pressure vector, $C$ is the coupling matrix between fluid and structure, and $\rho$ is the fluid density.

Problem (7) is known to have real eigenvalues [10], but since it is not symmetric AMLS as introduced in Section 2 is not applicable.

Kropp et al. [7, 11] suggested to solve the eigenvalue problems $K_s\phi_s = \omega_s M_s \phi_s$ for the structure and $K_f\phi_f = \omega_f M_f \phi_f$ for the fluid by symmetric AMLS independently, and to project problem (7) to the subspace spanned by $\begin{pmatrix} \Phi_s \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \Phi_f \end{pmatrix}$, where the columns of $\Phi_s$ and $\Phi_f$ are the eigenmodes of the the structure and fluid eigenproblem, respectively, which do not exceed a given cut off frequency. Thus they obtain a projected eigenproblem

$$\begin{pmatrix} \Omega_s & \Phi_s^T C \Phi_f \\ 0 & \Omega_f \end{pmatrix} \begin{pmatrix} \tilde{u}_s \\ \tilde{p}_f \end{pmatrix} = \lambda \begin{pmatrix} \Phi_s^T M_s \Phi_s & 0 \\ -\rho \Phi_f^T C^T \Phi_s & \Phi_f^T M_f \Phi_f \end{pmatrix} \begin{pmatrix} \tilde{u}_s \\ \tilde{p}_f \end{pmatrix} \tag{8}$$

of the same structure as the original problem (7), but of much smaller dimension. Note, that the coupling is not taken into consideration in the reduction process but only in the projection of the eigenproblem.

A different approach was considered in [5]. Eliminating $p_f$ in (7) one obtains the rational eigenvalue problem

$$K_s u_s = \lambda M_s u_s + \lambda \rho C (K_f - \lambda M_f)^{-1} C^T u_s. \tag{9}$$

Applying AMLS to the symmetric matrix pencil $(K_s, M_s)$ and applying all transformations and projections to the coupling matrix $C$ as well one obtains a rational eigenvalue problem of the same structure as (9) of much smaller dimension. This approach suffers the same weakness as the approach above that the coupling is not included into the reduction process.

## 4 A structure preserving version of AMLS

In this section we propose a modified AMLS algorithm for solving the fluid-solid vibration problem (7) in order to capture the interaction of fluid and solid in a more appropriate way.

Similar to the AMLS method for symmetric problem the joint graph of $K := \begin{pmatrix} K_s & C \\ 0 & K_f \end{pmatrix}$ and $M := \begin{pmatrix} M_s & 0 \\ -\rho C^T & M_f \end{pmatrix}$ is substructured recursively on several levels, but differently from the approach of Kropp and Heiserer the coupling matrix is incorporated into the substructuring process. Hence, any substructure may consist solely of degrees of freedom from the fluid or from the solid, or caused by the coupling matrix it may contain degrees of freedom of both types.

The crucial point is to modify the AMLS algorithm such that the structure of (7) is preserved for the reduced problem, and all eigenvalues are still real.

Pure fluid or solid substructures obviously can be treated in the same way as in the symmetric AMLS method, i.e. they can be decoupled in the stiffness matrix by Gaussian elimination and reduced by modal truncation. We now describe a typical general reduction step.

After a couple of reduction steps for problem (7) one arrives at the following matrices where the unknowns have been reordered appropriately, and $\rho$ has been set to 1 to save some space

$$\begin{pmatrix} K_{pp} & 0 & K^f_{p\ell} & 0 & K^f_{pi} & 0 & K^f_{pj} \\ K^s_{\ell p} & K^s_{\ell\ell} & C_\ell & K^s_{\ell i} & C_{\ell i} & K^s_{\ell j} & C_{\ell j} \\ 0 & 0 & K^f_{\ell\ell} & 0 & K^f_{\ell i} & 0 & K^f_{\ell j} \\ K^s_{ip} & K^s_{i\ell} & C_{i\ell} & K^s_{ii} & C_i & K^s_{ij} & C_{ij} \\ 0 & 0 & K^f_{i\ell} & 0 & K^f_{ii} & 0 & K^f_{ij} \\ K^s_{jp} & K^s_{j\ell} & C_{j\ell} & K^s_{ji} & C_{ji} & K^s_{jj} & C_j \\ 0 & 0 & K^f_{j\ell} & 0 & K^f_{ji} & 0 & K^f_{jj} \end{pmatrix}, \begin{pmatrix} K_{pp} & M^s_{p\ell} & M^f_{p\ell} & M^s_{pi} & M^f_{pi} & M^s_{pj} & M^f_{pj} \\ M^s_{\ell p} & M^s_{\ell\ell} & 0 & M^s_{\ell i} & 0 & M^s_{\ell j} & 0 \\ M^f_{\ell p} & -C^T_\ell & M^f_{\ell\ell} & -C^T_{i\ell} & M^f_{\ell i} & -C^T_{j\ell} & M^f_{\ell j} \\ M^s_{ip} & M^s_{i\ell} & 0 & M^s_{ii} & 0 & M^s_{ij} & 0 \\ M^f_{ip} & -C^T_{\ell i} & M^f_{i\ell} & -C^T_i & M^f_{ii} & -C^T_{ji} & M^f_{ij} \\ M^s_{jp} & M^s_{j\ell} & 0 & M^s_{ji} & 0 & M^s_{jj} & 0 \\ M^f_{jp} & -C^T_{\ell j} & M^f_{j\ell} & -C^T_{ij} & M^f_{ji} & -C^T_j & M^f_{jj} \end{pmatrix}. \tag{10}$$

Here $p$ denotes the degrees of freedom obtained in the reduction steps on previous levels, $\ell$ collects the degrees of freedom to be handled in the current step, $i$ corresponds to the index set of parent substructures, and $j$ denotes interface variables of even coarser levels. A superscript $s$ denotes the structure part, and $f$ the fluid part

of the model. Notice that the $K$ and $M$ part of the lower-right $6\times 6$ block (i.e. the part which is obtained if the $C$ blocks are replaced by 0) are symmetric and positive definite.

Annihilating $K^s_{\ell k}$ and $K^f_{\ell k}$, $k\in\{i,j\}$ by symmetric block Gaussian elimination one obtains

$$\begin{pmatrix} K_{pp} & 0 & K^f_{p\ell} & 0 & \tilde K^f_{pi} & 0 & \tilde K^f_{pj}\\ K^s_{\ell p} & K^s_{\ell\ell} & C_\ell & 0 & \tilde C_{\ell i} & 0 & \tilde C_{\ell j}\\ 0 & 0 & K^f_{\ell\ell} & 0 & 0 & 0 & 0\\ \tilde K^s_{ip} & 0 & \tilde C_{i\ell} & \tilde K^s_{ii} & \tilde C_i & \tilde K^s_{ij} & \tilde C_{ij}\\ 0 & 0 & 0 & 0 & \tilde K^f_{ii} & 0 & \tilde K^f_{ij}\\ \tilde K^s_{jp} & 0 & \tilde C_{j\ell} & \tilde K^s_{ji} & \tilde C_{ji} & \tilde K^s_{jj} & \tilde C_j\\ 0 & 0 & 0 & 0 & \tilde K^f_{ji} & 0 & \tilde K^f_{jj} \end{pmatrix}, \begin{pmatrix} K_{pp} & M^s_{p\ell} & M^f_{p\ell} & \tilde M^s_{pi} & \tilde M^f_{pi} & \tilde M^s_{pj} & \tilde M^f_{pj}\\ M^s_{\ell p} & M^s_{\ell\ell} & 0 & \tilde M^s_{\ell i} & 0 & \tilde M^s_{\ell j} & 0\\ M^f_{\ell p} & -C^T_\ell & M^f_{\ell\ell} & -\tilde C^T_{i\ell} & \tilde M^f_{\ell i} & -\tilde C^T_{j\ell} & \tilde M^f_{\ell j}\\ \tilde M^s_{ip} & \tilde M^s_{i\ell} & 0 & \tilde M^s_{ii} & 0 & \tilde M^s_{ij} & 0\\ \tilde M^f_{ip} & -\tilde C^T_{\ell i} & \tilde M^f_{i\ell} & -\tilde C^T_i & \tilde M^f_{ii} & -\tilde C^T_{ji} & \tilde M^f_{ij}\\ \tilde M^s_{jp} & \tilde M^s_{j\ell} & 0 & \tilde M^s_{ji} & 0 & \tilde M^s_{jj} & 0\\ \tilde M^f_{jp} & -\tilde C^T_{\ell j} & \tilde M^f_{j\ell} & -\tilde C^T_{ij} & \tilde M^f_{ji} & -\tilde C^T_j & \tilde M^f_{jj} \end{pmatrix} \tag{11}$$

where a tilde indicates that the associated matrix that has been modified in the elimination.

The next step is to solve the substructure eigenvalue problem

$$\begin{pmatrix} K^s_{\ell\ell} & C_\ell\\ 0 & K^f_{\ell\ell}\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \omega\begin{pmatrix} M^s_{\ell\ell} & 0\\ -C^T_\ell & M^f_{\ell\ell}\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}. \tag{12}$$

which has real eigenvalues and eigenvectors due to the matrix structure described above.

If $\left(\phi^T,\psi^T\right)^T$ is a right eigenvector of (12) corresponding to the positive eigenvalue $\omega$, then it is easily seen that $\left(\phi^T,\frac{1}{\omega}\psi^T\right)^T$ is a left eigenvector corresponding to $\omega$.

We assume that the eigenvectors are normalized such that

$$\left(\phi_i^T,\tfrac{1}{\omega_i}\psi_i^T\right)\begin{pmatrix} M^s_{\ell\ell} & 0\\ -C^T_\ell & M^f_{\ell\ell}\end{pmatrix}\begin{pmatrix}\phi_j\\ \psi_j\end{pmatrix} = \delta_{ij} \tag{13}$$

As in standard AMLS for symmetric problems we neglect eigenvectors corresponding to eigenvalues exceeding a given cut-off-frequency. Let the columns of $\Phi_\ell$ and $\Psi_\ell$ be the structure and fluid part of the kept eigenvectors, respectively, and let the diagonal matrix $\Omega$ contain the according eigenvalues. Multiplying the matrices in (11) by

$$\begin{pmatrix} I & 0 & 0 & 0 & 0\\ 0 & \Phi_\ell^T & \Omega^{-1}\Psi_\ell^T & 0 & 0\\ 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & 0 & I\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} I & 0 & 0 & 0\\ 0 & \Phi_\ell & 0 & 0\\ 0 & \Psi_\ell & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{pmatrix} \tag{14}$$

from the left and right, respectively, one finally ends up with the reduced matrices in step $\ell$

$$\begin{pmatrix} K_{pp} & \tilde{K}_{p\ell} & 0 & \tilde{K}^f_{pi} & 0 & \tilde{K}^f_{pj} \\ \tilde{K}_{\ell p} & \Omega & 0 & \tilde{C}_{\ell i} & 0 & \tilde{C}_{\ell j} \\ \tilde{K}^s_{ip} & \tilde{K}_{i\ell} & \tilde{K}^s_{ii} & \tilde{C}_i & \tilde{K}^s_{ij} & \tilde{C}_{ij} \\ 0 & 0 & 0 & \tilde{K}^f_{ii} & 0 & \tilde{K}^f_{ij} \\ \tilde{K}^s_{jp} & \tilde{K}_{j\ell} & \tilde{K}^s_{ji} & \tilde{C}_{ji} & \tilde{K}^s_{jj} & \tilde{C}_j \\ 0 & 0 & 0 & \tilde{K}^f_{ji} & 0 & \tilde{K}^f_{jj} \end{pmatrix}, \begin{pmatrix} M_{pp} & \tilde{M}_{p\ell} & \tilde{M}^s_{pi} & \tilde{M}^f_{pi} & \tilde{M}^s_{pj} & \tilde{M}^f_{pj} \\ \tilde{M}_{\ell p} & I & \tilde{M}^s_{\ell i} & \tilde{M}^f_{\ell i} & \tilde{M}^s_{\ell j} & \tilde{M}^f_{\ell j} \\ \tilde{M}^s_{ip} & \tilde{M}^s_{i\ell} & \tilde{M}^s_{ii} & 0 & \tilde{M}^s_{ij} & 0 \\ \tilde{M}^f_{ip} & \tilde{M}^f_{i\ell} & -\tilde{C}^T_i & \tilde{M}^f_{ii} & -\tilde{C}^T_{ji} & \tilde{M}^f_{ij} \\ \tilde{M}^s_{jp} & \tilde{M}^s_{j\ell} & \tilde{M}^s_{ji} & 0 & \tilde{M}^s_{jj} & 0 \\ \tilde{M}^f_{jp} & \tilde{M}^f_{j\ell} & -\tilde{C}^T_{ij} & \tilde{M}^f_{ji} & -\tilde{C}^T_j & \tilde{M}^f_{jj} \end{pmatrix} \tag{15}$$

Obviously the lower right $4 \times 4$ block of these matrices has the same structure as the lower right $6 \times 6$ block of the matrices in (10) demonstrating that the structure of the matrices is preserved in the AMLS reduction. Decoupling and reducing all coarser substructures in the same way one finally arrives at the projected eigenvalue problem.

## 5 Numerical results

To evaluate the performance of the AMLS method above we consider the free vibrations of a tube bundle immersed in a fluid (cf. [3]). Discretizing by linear Lagrangian elements one obtains an eigenvalue problem (7) with 143082 degrees of freedom which has already been examined with a variant of AMLS for the rational eigenproblem (9) in [5]. The problem was partitioned into 2045 substructures on 11 levels.

In this problem the structure and the fluid are coupled very strongly, i.e. the resonance frequencies of the uncoupled problems do not approximate the resonance frequencies of the coupled problem well. For instance, there are 12 eigenfrequencies of the fluid which are less than the smallest structure eigenvalue $\omega_s$ whereas the coupled system has 18 eigenvalues not exceeding $\omega_s$.

Applying AMLS for symmetric definite problems, it is usually sufficient to choose the cut-off-frequency as a small multiple of the maximum desired eigenvalue. For coupled problems, this cut-off-frequency has to be increased to get acceptable relative errors of the eigenvalue approximation. In our example we determined all eigenvalues less than 3.5, and performed the calculation with a cut-off-frequency of $\omega_c = 100$.

Figure 1 shows the relative errors of the eigenvalues obtained by AMLS incorporating the coupling into the reduction process (crosses) as compared to relative errors achieved by the AMLS approach in [7] where the coupling is included only into the projected problem (circles). As reference value we used the results received from the software package ARPACK [8].

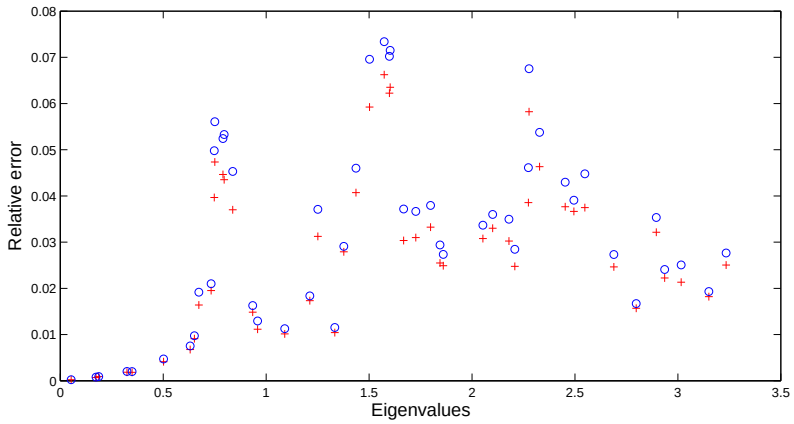


**Fig. 1** Relative error for the eigenvalue approximations received from the method described above (crosses) and the method described in [7] (circles)

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