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# RESTARTING PROJECTION METHODS FOR RATIONAL EIGENPROBLEMS

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**Abstract.** For nonlinear eigenvalue problems $T(\lambda)x = 0$ satisfying a minmax characterization of its eigenvalues iterative projection methods combined with safeguarded iteration are suitable for computing all eigenvalues in a given interval. Such methods hit their limitation if a large number of eigenvalues is required. In this paper we discuss restart procedures which are able to cope with this problem, and we evaluate them for a rational eigenvalue problem governing vibrations of a fluid-solid structure.

**Key words:** nonlinear eigenvalue problem, iterative projection method, Arnoldi method, minmax characterization, restart, fluid-solid structure

## 1. Introduction

We consider the nonlinear eigenvalues problem

$$T(\lambda)x = 0, \tag{1.1}$$

where $T(\lambda) \in \mathbb{C}^{n\times n}$ is a family of large and sparse Hermitean matrices for every $\lambda$ in an open real interval $J$. For this type of problems iterative projection methods are very efficient (cf. [4, 6] an the literature given therein). In particular, if the eigenvalues of $T(\cdot)$ satisfy a minmax characterization, all eigenvalues can be determined one after the other in a safe way. However, this approach hits it limitations if a large number of eigenvalues or eigenvalues in the interior of the spectrum of (1.1) is needed. In this case one has to project the problem under consideration onto a sequence of search spaces of growing dimensions requiring an excessive amount of storage and computing time. In [2] we presented a new restart technique which projects problem (1.1) only to search spaces of limited dimension. Here we generalize this approach to

rational eigenvalue problem governing free vibrations of fluid-solid structures. Our presentation is restricted to the Arnoldi method, but the local restart technique applies to any other iterative projection method.

The paper is organized as follows. Section 2 outlines the variational characterization of eigenvalues for nonlinear and nonoverdamped eigenproblems and the safeguarded iteration method, and Section 3 recalls the Arnoldi method for sparse, symmetric, and nonlinear eigenproblems. In Section 4 we present the local restart technique. In particular we discuss the problem of spurious eigensolutions. An example of a rational eigenproblem in Section 5 demonstrates the efficiency of the new restart procedure.

## 2. Variational characterization of eigenvalues

In this section we recall conditions under which the eigenvalues of $T(\cdot)$ can be characterized as minmax values of a Rayleigh functional. Let $J \subset \mathbb{R}$ be an open interval which may be unbounded, and assume that $T(\lambda) \in \mathbb{C}^{n\times n}$ is a family of Hermitean matrices. Suppose that for every $x \in \mathbb{C}^n \setminus \{0\}$ the real equation

$$f(\lambda, x) := x^H T(\lambda)x = 0 \tag{2.1}$$

has at most one solution $\lambda \in J$. Then equation (2.1) defines a functional $p$ on some subset $D \subset \mathbb{C}^n$ which obviously generalizes the Rayleigh quotient for linear pencils $T(\lambda) = \lambda B - A$, and which is called the Rayleigh functional.

We assume further

$$(\lambda - p(x))f(\lambda, x) > 0 \quad \text{for every } x \in D \text{ and } \lambda \in J \setminus \{p(x)\}$$

generalizing the definiteness condition for linear pencils.

Under these conditions a minmax principle for the nonlinear eigenproblem (1.1) was proved in [7] if the eigenvalues are enumerated appropriately. A value $\lambda \in J$ is an eigenvalue of (1.1) if and only if $\mu = 0$ is an eigenvalue of the matrix $T(\lambda)$, and by Poincaré's maxmin principle there exists $m \in \mathbb{N}$ with

$$0 = \max_{\dim V = m} \min_{x\in V,\ x\neq 0} \frac{x^H T(\lambda)x}{\|x\|^2}.$$

Then we assign this $m$ to $\lambda$ as its number and calls $\lambda$ an $m$th eigenvalue of problem (1.1).

Under the assumptions above it was shown in [7] that for every $m \in \{1, \ldots, n\}$ problem (1.1) has at most one $m$th eigenvalue in $J$, which can be characterized by

$$\lambda_m = \min_{\dim V = m, D\cap V\neq\emptyset} \sup_{v\in D\cap V} p(v). \tag{2.2}$$

The minimum is attained by the invariant subspace of $T(\lambda_m)$ corresponding to its $m$ largest eigenvalues, and the supremum is attained by any eigenvector of $T(\lambda_m)$ corresponding to $\mu = 0$.

The enumeration of eigenvalues and the fact that the eigenvectors of (1.1) are the stationary vectors of the Rayleigh functional suggests the Algorithm 1 called safeguarded iteration for computing the $m$th eigenvalue.

**Algorithm 1** Safeguarded iteration

1: Start with an approximation $\sigma_1$ to the $m$th eigenvalue of (1.1)
2: **for** $k = 1, 2, \ldots$ until convergence **do**
3: Compute an eigenvector $x_k$ corresponding to the $m$-largest eigenvalue of $T(\sigma_k)$
4: Solve $x_k^H T(\sigma_{k+1}) x_k = 0$ for $\sigma_{k+1}$
5: **end for**

Safeguarded iteration has the following convergence properties (cf. [5]):

(i) If $\lambda_1 := \inf_{x \in D} p(x) \in J$ and $x_1 \in D$ then the safeguarded iteration converges globally to $\lambda_1$.
(ii) If $\lambda_m \in J$ is a $m$th eigenvalue of (1.1) which is simple, then the safeguarded iteration converges locally and quadratically to $\lambda_m$.

The safeguarded iteration is certainly not capable to solve large and sparse nonlinear eigenvalue problems. However, it is well suited as an inner iteration in a projection method, especially if one is interested in particular eigenvalues, e.g. in an interval. Since it aims at an eigenvalue with a specific number missing out of eigenvalues becomes less likely.

## 3. Iterative projection methods

Iterative projection methods like Lanczos, Arnoldi or Jacobi-Davidson are very efficient for sparse linear eigenvalue problems, and so are their generalizations to nonlinear ones. A typical example is the nonlinear Arnoldi method in Algorithm 2, where we assume that problem (1.1) is Hermitean, and the eigenvalues can be enumerated according to Section 2.

**Algorithm 2** Nonlinear Arnoldi Method

1: Start with initial basis $V$, $V^H V = I$; $m = 1$
2: Determine preconditioner $M \approx T(\sigma)^{-1}$, $\sigma$ close to first wanted eigenvalue
3: **while** $m \leq$ number of wanted eigenvalues **do**
4: Compute $m$ smallest eigenvalue $\mu$ and corresponding eigenvector $y$ of the projected problem $T_V(\mu)y := V^H T(\mu) V y = 0$ by safeguarded iteration
5: Determine Ritz vector $u = Vy$ and residual $r = T(\mu)u$
6: **if** $\|r\|/\|u\| < \epsilon$ **then**
7: Accept approximate eigenpair $\lambda_m = \mu$, $x_m = u$; increase $m \leftarrow m + 1$
8: Choose new shift $\sigma$ and determine preconditioner $M \approx T(\sigma)^{-1}$ if indicated
9: Restart if necessary
10: Find approximation $(\mu, u)$ to next eigenpair; determine residual $r = T(\mu)u$
11: **end if**
12: $v = Mr$
13: $v = v - VV^H v$, $\tilde{v} = v/\|v\|$
14: Reorthogonalize $\tilde{v}$ if necessary
15: Expand subspace: $V = [V, \tilde{v}]$
16: **end while**

There are many details that have to be considered when implementing the Arnoldi method according to Algorithm 2 concerning the choice of the initial basis, when to change and how to choose the preconditioner, when and how to restart. A detailed discussion is contained in [6]. Here we concentrate on the initialization and on restarts for symmetric problems allowing a minmax characterization of their eigenvalues.

If $T(\lambda)$ is a family of Hermitean matrices allowing a minmax characterization of its eigenvalues in an open interval $J$, and if the columns of $V \in \mathbb{C}^n$ form a basis of the current search space $\mathcal{V}$ of $\mathbb{C}^n$, then the projected problem

$$T_V(\lambda)y := V^H T(\lambda)Vy = 0 \tag{3.1}$$

inherits this property, i.e. its eigenvalues in $J$ are minmax values of the restriction of the Rayleigh functional $p$ of $T(\cdot)$ to $D \cap \mathcal{V}$, although in general the numeration of the eigenvalues of the original problem and the projected problem will differ.

If $J$ contains a first eigenvalue $\lambda_1 = \min_{x \in D} \; p(x)$, then the safeguarded iteration for (3.1) converges globally for any initial vector $x \in \mathcal{V} \cap D$ to the smallest eigenvalue of (3.1). If $x_j$ denotes an eigenvector corresponding to the $j$th eigenvalue $\lambda_j$ of (1.1), and if $x_j \in \mathcal{V}$ for $j = 1, \ldots, k$, then $\lambda_j$ is a $j$th eigenvalue of the projected problem (3.1), as well. Hence, expanding the search space $\mathcal{V}$ iteratively, and determining the $(k+1)$TH eigenvalue of the projected problems, one gets a sequence of upper bounds of $\lambda_{k+1}$ which (hopefully) converges to $\lambda_{k+1}$. Thus, the eigenvalues of (1.1) can be determined one after the other by the nonlinear Arnoldi algorithm.

As the subspace expands in the course of the algorithm the increasing storage or the computational cost for solving the projected eigenvalue problems may make it necessary to restart the algorithm and purge some of the basis vectors. Restarting with a subspace $\mathcal{V}$ which contains the already converged eigenvectors $x_1, \ldots, x_k$ then obviously keeps the numeration of the eigenvalues, and we can continue as above to determine the subsequent eigenpairs.

## 4. Local restarts

The restart strategy described in the last section hits it limitations if a large number of eigenvalues or a set of some subsequent eigenvalues in the interior of the spectrum is required. In order to preserve the numbering the dimension of the search space has to be at least as large as the number of eigenvalues in $J$ preceding the sought one. Therefore the size of the projected problem is growing with the number of the wanted eigenvalue, which results in increasing time consumed by the nonlinear solver and increasing storage requirements. To overcome these difficulties we proposed an adjusted version of a local numbering [2], which does not require to include the entire set of preceding eigenvectors into the search subspace after a restart.

Assume that we are given an eigenvalue $\hat{\lambda} \in J$, which we call an anchor, and a corresponding eigenvector $\hat{x}$. Let $\mathcal{V}$ be a subspace of $\mathbb{C}^n$ that contains

**Algorithm 3** Restart Framework

**Require:** $(\lambda_i, x_i)$ an (approximate) eigenpair of $T(\cdot)$
**Require:** $v_1$ an approximation to $x_{i+1}$
$V = [x_i, v_1]$; $j = 1$;
**while** Restart condition not satisfied **do**
  **repeat**
    Determine largest eigenvalues $\mu_1(\lambda_i) \geq \cdots \geq \mu_k(\lambda_i) > 0 \geq \mu_{k+1}(\lambda_i)$ of projected problem (3.1)
    Set $\ell := k$ if $\mu_k \leq -\mu_{k+1}$, and else $\ell := k+1$
    Compute $(\ell + j)$th eigenpair $(\tilde{\lambda}_{\ell+j}, y_{\ell+j})$ of $T_V(\cdot)$
    Expand $V$ aiming at $(\lambda_{\ell+j}, x_{\ell+j})$
  **until** Eigenpair $(\tilde{\lambda}_{\ell+j}, Vy_{l+j}) =: (\lambda_{i+j}, x_{i+j})$ converged
  j = j+1;
**end while**

$\hat{x}$, and let the columns of $V$ form a basis of $\mathcal{V}$. Then $\hat{\lambda}$ is also an eigenvalue of the projected problem

$$T_V(\lambda) := V^H T(\lambda) V y = 0. \tag{4.1}$$

Moreover, $T_V(\cdot)$ satisfies the conditions of the minmax characterization. Thus, we can assign to $\hat{\lambda}$ a local number $\ell = \ell(\mathcal{V})$ in the following way: $\hat{\lambda}$ is an $\ell$th eigenvalue of problem (4.1) if $\mu(\hat{\lambda}) = 0$ is the $\ell$ largest eigenvalue of the linear problem $V^H T(\hat{\lambda}) V y = \mu(\hat{\lambda}) y$.

Starting with $\mathcal{V} =: \mathcal{V}_0$ we determine approximations to the eigenvalue subsequent to the anchor $\hat{\lambda}$, projecting problem (1.1) to a sequence of subspaces $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots$ which are generated in course of the Arnoldi method aiming at the $(\ell(\mathcal{V}_k) + 1)$th eigenvalue in the $k$th iteration step. Explicitly stating the dependence of $\ell$ on $\mathcal{V}_k$ we emphasize that the number $\ell(\mathcal{V}_k)$ of the anchor may change in the course of the iteration.

After convergence we may continue the iterative projection method aiming at the $(\ell(\mathcal{V}_k) + 2)$th eigenvalue or we may replace the anchor by the newly converged eigenpair. Since the current search space contains useful information about further eigenvalues it is advisable to continue expanding the search spaces until the convergence has become too slow or the dimension exceeds a given bound. This leads to the restart framework in Algorithm 3.

In the course of the computation it may happen that the algorithm converges to an eigenvalue twice, i.e. it returns $\lambda_i < \lambda_{i+1} < \cdots < \lambda_{i+k} \approx \lambda_{i+k+1}$ for some $k \geq 1$. Then two possibilities arise:

(i) the eigenvalue is a multiple (at least double) eigenvalue;
(ii) the algorithm converged to a single eigenvalue twice.

If $\lambda_{i+k}$ is not a multiple eigenvalue (i.e. if the angle between $x_{i+k}$ and $x_{i+k+1}$ is close to 0), then for the current search space $\mathcal{V}$ the projected problem (4.1) possesses an additional eigenvalue $\theta \in (\lambda_i, \lambda_{i+k})$ such that $\theta \neq \lambda_{i+j}$ for $j = 0, \ldots, k$. Therefore the local number of $\lambda_{i+k}$ is raised by 1, and $\lambda_{i+k}$ is accepted as an $(i + k + 1)$th eigenvalue. This may have happened for one of the following two reasons:

— An eigenvalue of (1.1) in the interval $(\lambda_i, \lambda_{i+k})$ might have been missed out because the corresponding eigenvector $\tilde{x}$ was not sufficiently present in the initial search space $\mathrm{span}\{x_i, v_1\}$ and might not have been amplified sufficiently in the course of the expansions of $\mathcal{V}$ until computing $\lambda_{i+k+1}$. Afterwards the component of $\tilde{x}$ in the search space $\mathcal{V}$ has increased and has become large enough to produce the additional eigenvalue approximation $\theta \in (\lambda_i, \lambda_{i+k})$.
— It might be the case that no eigenvalue of (1.1) is missing in $(\lambda_i, \lambda_{i+k})$ but the newly produced eigenvalue of the projected problem (4.1) is a spurious one, i.e. is a linear combination of eigenvectors of (1.1) corresponding to eigenvalues less than $\lambda_i$ and of eigenvectors corresponding to eigenvalues greater than $\lambda_{i+k}$.

In both cases we determine the additional eigenvalue $\theta$ and its local number $\ell+j$, and we expand the search space $\hat{\mathcal{V}}$ aiming at $(\theta, x_\theta)$, where $x_\theta$ denotes the Ritz vector corresponding to $\theta$. Then by the minmax principle all eigenvalues of the projected problem

$$T_{\hat{V}}(\lambda)\hat{y} = 0 \tag{4.2}$$

are less than or equal to the corresponding ones of $T_V(\lambda)y = 0$, and either problem (4.2) has exactly $k+1$ eigenvalues $\lambda_i, \dots, \lambda_{i+k} \in [\lambda_i, \lambda_{i+k}]$ (i.e. the additional eigenvalue has left the interval of interest) or there are still $k+2$ eigenvalues $\lambda_i, \dots, \lambda_{i+k}, \hat{\theta} \in [\lambda_i, \lambda_{i+k}]$, and it holds $\hat{\theta} \le \theta$. In the latter case we repeat the expansion of the subspace until the sequence of additional eigenvalues has been moved out of the interval $[\lambda_i, \lambda_{i+k}]$ or has converged to a previously missed out regular eigenvalue. We then adjust the enumeration of the eigenvalues and continue the iterative projection method.

## 5. Restarts for rational eigenvalue problems

We consider a rational eigenvalue problem

$$Kx = \lambda Mx + \sum_{j=1}^{k} \frac{\lambda}{\sigma_j - \lambda} C_j C_j^T x \tag{5.1}$$

where $K, M \in \mathbb{R}^{n\times n}$ are symmetric and positive definite, $C_j \in \mathbb{R}^{n\times r_j}$ are matrices of small rank $r_j$, and $0 < \sigma_1 < \sigma_2 < \cdots < \sigma_k$ are given poles. Problems of this type arise for example as finite element models of rational eigenproblems governing free vibrations of tube bundles immersed in a slightly compressible fluid [3].

It is easily seen that in each of the intervals $J_j := (\sigma_{j-1}, \sigma_j)$, $j = 1, \dots, k+1$ with $\sigma_0 = 0$, $\sigma_{k+1} = \infty$, problem (5.1) satisfies the conditions of the minmax characterization (2.2). In $J_1$ the eigenvalues are numerated in the natural way. For the following intervals the numeration follows from Theorem 1 which was proved in [5].

**Theorem 1.** *Let $n_j$ be the number of eigenvalues $\lambda_i$ of the reduced linear eigenvalue problem*

$$\Big(K+\sum_{i=1}^{j-1}\frac{\sigma_j}{\sigma_j-\sigma_i}C_iC_i^T\Big)x=\lambda\Big(M+\sum_{i=j+1}^{k}\frac{1}{\sigma_i-\sigma_j}C_iC_i^T\Big)x,\quad C_jx=0$$

*satisfying $\lambda_i\le\sigma_j$.*

*Then the rational eigenvalue problem (5.1) has exactly $n_{j+1}-n_j+r_{j+1}$ eigenvalues in $(\sigma_j,\sigma_{j+1}]$ enumerated by $n_j+1,n_j+2,\dots,n_{j+1}+r_{j+1}$.*

This theorem suggests a global strategy how to determine all eigenvalues in a given interval $I\subset\mathbb{R}_+$ by the nonlinear Arnoldi method similar to the approach at the end of Section 3.

For $J_1=(0,\sigma_1)$ the infimum of the Rayleigh functional is contained in $J_1$, and due to the global convergence of the safeguarded iteration we can start with any vector $v$ such that $p(v)\in J_1$, and compute the eigenvalues in $J_1$ one after the other until the method leaves the interval $J_1$.

Assume that we have already determined all eigenvalues in $J_1,\dots J_j$, and let $N_j$ be the number (according to the enumeration in Section 2 for problem (5.1) in $J_j$) of the largest eigenvalue $\lambda_{N_j}$ found in $J_j$. To start the Arnoldi method for $J_{j+1}:=(\sigma_j,\sigma_{j+1})$ we choose $\hat\mu=\sigma_j+\varepsilon$, $\varepsilon>0$ small, and determine the eigenvalues of the linear problem

$$\Big(K+\sum_{i=1}^{j}\frac{\mu}{\mu-\sigma_i}C_iC_i^T\Big)x=\lambda\Big(M+\sum_{i=j+1}^{k}\frac{1}{\sigma_i-\mu}C_iC_i^T\Big)x,\qquad(5.2)$$

for $\mu=\hat\mu$ which are less than $\hat\mu$. We assume that these are $\tilde n_j$. If $\tilde n_j+r_j=N_j$ then all eigenvalues in $(0,\sigma_j)$ have been found. No eigenvalue exist in $(\sigma_j,\hat\mu)$, and we can start the Arnoldi method for $J_{j+1}$ with an orthonormal basis $V_{j+1}$ of the eigenspace of problem (5.2) corresponding to the $n_j+1$ smallest eigenvalues. Otherwise we have to explore the interval $(\lambda_{N_j},\hat\mu)$ for further eigenvalues.

The method can be restarted if the search space has grown to large where the current initial basis $V_j$ has to be complemented by the eigenvectors corresponding to eigenvalues found in the current interval $J_j$ so far. The method (which we call GLOBAL 1) is safe, but costly if the number of poles $\sigma_j$ less than $\sup I$ (i.e. the number of linear eigenvalue problems (5.2) to be solved) is large or if there are many eigenvalues in $(0,\sup I)$.

Another variant (called GLOBAL 2) of the iterative projection method can be devised on the basis of Theorem 1. Again we address one interval after the other and within each interval one eigenvalue after the other adapting the number of the eigenvalue for each interval as given by Theorem 1, i.e. every time when a pole is being crossed the number of the eigenvalue is reduced by the rank $r_j$ of the matrix $C_j$. This variant fully utilizes the information acquired while computing the eigenvalues in the former intervals, since the computation continuous over multiple intervals and no linear problems are being solved. However, there is no guarantee that all eigenvalues in the interval

treated so far have been found. Moreover, no restart strategy is implemented and therefore the method is not suitable for a large number of eigenvalues.

This can be cured by combining the two variants in a way that we compute eigenvalues in several intervals corresponding to method GLOBAL 2, and restart corresponding to method GLOBAL 1 as soon as the search subspace has grown to large. Obviously, this method (called GLOBAL 3) can be restarted at any point $\hat{\mu}$. On restart we can check again, whether eigenvalues have been missed in the computation so far.

Finally, the local restart strategy can be applied to the rational eigenvalue problem in the following way. Let $\hat{\lambda}$ denote an anchor, and $\hat{x}$ the corresponding eigenvector of (1.1), and let $\ell$ denote the local number of the anchor in the current search subspace $\mathcal{V}$. According to Theorem 1 the number of an eigenvalue is not unique on the real axis and therefore always needs to be considered with the interval containing the eigenvalue. Therefore we denote by $\ell + k_j$ the number of the eigenvalue $\lambda_{\ell+k_j}$ in the interval $(\sigma_j, \sigma_{j+1})$.

As long as the anchor $\hat{\lambda}$ and the currently iterated eigenvalue $\lambda_{\ell+k_j}$ are in the same interval $(\sigma_j, \sigma_{j+1})$ the local enumeration holds exactly as in Section 4. However, if a pole has been crossed, i.e. $\hat{\lambda} \in (\sigma_j, \sigma_{j+1})$ and $\lambda_{\ell+k_{j+1}} \in (\sigma_{j+1}, \sigma_{j+2})$ the situation is different.

According to Theorem 1 if $\lambda_{\ell+k_j}$ is the largest eigenvalue smaller than $\sigma_{j+1}$ then the number of the next eigenvalue is $\ell + k_{j+1} = \ell + k_j + 1 - r_{j+1}$, where the offset $r_{j+1}$ is equal to the rank of the corresponding matrix $C_{j+1}$. For the iterative projection methods it means that at the $l$th step of the iteration with the search subspace $\mathcal{V}_l$, the offset for the projected problem is $r^l_{j+1} = \operatorname{rank} V_l^H C_{j+1}$. This is bad news, since the offset $r^l_{j+1}$ can vary in the course of the iteration. This can be helped in three ways:

— The simplest but unsatisfactory solution is to reset the anchor every time a pole is being crossed. However, such a procedure violates the condition that the anchor is an eigenvalue of (1.1) since the anchor would have to be set to an approximation to the eigenvalue obtained so far. In this way missing out eigenvalues becomes more probable;
— A better solution is to recompute the offset $r^l_{j+1}$ on every iteration as long as $r^l_{j+1} = \operatorname{rank} V_l^H C_{j+1} < \operatorname{rank} C_{j+1}$. Once $\operatorname{rank} V_l^H C_{j+1} = \operatorname{rank} C_{j+1}$ the offset remains the same until we discard a part of the subspace $\mathcal{V}_l$. However, in the worst case we have to compute the rank at every iteration;
— In our view the best solution is to introduce the absent vectors by including $\operatorname{span} C_{j+1}$ corresponding to the $j+1$th pole lying between the anchor and the currently iterated eigenvalue. Clearly, this strategy is not restricted to the two neighboring intervals but can be used over multiple neighboring intervals.

The presented strategy is flexible to restart whenever necessary keeping the size of the basis moderate. Furthermore the restart does not require expensive solving of a generalized linear eigenvalue problem (5.2) of a dimension growing with the number of the eigenvalue.

The described local enumeration clearly does not interfere with the strategies of dealing with spectral pollution described in Section 3. The only issue

is to keep track of which intervals the anchor and the currently iterated eigenvalue are contained in and correspondingly adjust the offset of the currently iterated eigenvalue from the anchor. For other strategies of dealing with spectral pollution see [1].

## 6. Numerical experiments

In this section we evaluate four different variants of iterative projection method for the computation of the interior eigenvalues of (5.1). We consider a rational eigenproblem (5.1) of dimension $n = 36040$ with poles $\sigma_j := j$ and $\text{rank}C_j = 2$ for $j = 1, \ldots, 9$ which is a finite element model of an elliptic cavity with 9 immersed tubes. Details about the model can be found in [1].

As a benchmark we compute all the eigenvalues in the intervals $(0, 10)$, $(7, 10)$ and $(9, 10)$ by the four restart variants of the nonlinear Arnoldi iteration discussed in the last section. We report CPU times as obtained under MATLAB 2006a on a Intel Pentium D processor with 3.2 GHz and 4 GB RAM.

The first three lines of Table 1 show the total CPU times for computing all eigenvalues with strategy GLOBAL 1, the times consumed by the nonlinear solvers and the times necessary for the initialization. The initialization requires

**Table 1.** CPU times for GLOBAL strategies

| strategy | interval | total CPU [s] | nonlin. CPU [s] | init. CPU [s] |
|---|---|---|---|---|
| GLOBAL 1 | (0,10) | 511 | 12 | 157 |
| | (7,10) | 183 | 6.7 | 83.3 |
| | (9,10) | 65.8 | 2.6 | 32.2 |
| GLOBAL 2 | (0,10) | 320 | 128 | 3.1 |
| GLOBAL 3 | (7,10) | 89 | 11.2 | 36.7 |

30% to 50% of the total computing time since many intervals and eigenvalues are involved.

The total CPU times can be reduced considerably by GLOBAL 2 for the interval $(0, 10)$ since then with GLOBAL 1 a large number of initializations is necessary, and by GLOBAL 3 for the interval $(7, 10)$ since then it is not necessary to compute all eigenpairs in $(0, 7)$. Lines 4 and 5 demonstrate that this is actually the case. We gain about 40% of the total CPU time by GLOBAL 2 for $(0, 10)$ and about 50% of the CPU time by GLOBAL 3 for $(7, 10)$. Nevertheless, as we infer from the time consumed by the nonlinear solver in GLOBAL 2, this strategy is not suitable for the computation of a large number of eigenvalues since the search subspace is growing too large.

Finally, we apply the nonlinear Arnoldi method with local restarts. The corresponding computation times are shown in Table 2. Comparing to the

global variants we gain about 15% for the interval $(0, 10)$ and about 10% for $(9, 10)$ but we need more time for the interval $(7, 10)$. It means, that for large intervals where the search subspace would have grown large or for a single interval where the effects of crossing a pole do not arise, the locally restarted variant performs better. However, as long as the size of the search subspace including all preceding vectors is moderate and we encounter pole crossings the restarted sequential variant is superior since the global numbering holds and hence no spurious eigenvalues arise.

**Table 2.** Local restarts

| | total CPU [s] | nonlin. CPU [s] |
|---|---|---|
| (0,10) | 266 | 17 |
| (7,10) | 100 | 7.3 |
| (9,10) | 58 | 0.3 |

The main advantage of the local restart strategy is that we do not require all the eigenvectors to be in the search subspace. Thus we can initialize the problem by computing the eigenvalue with the smallest magnitude of the shifted linear problem (5.2). Moreover, in contrast to the first two variants we do not encounter orthogonality problems since the search subspace remains of moderate size.

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