

# A Genuinely Multi-dimensional Relaxation Scheme for Hyperbolic Conservation Laws

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## Abstract

A new genuinely multi-dimensional relaxation scheme is proposed. Based on a new discrete velocity Boltzmann equation, which is an improvement over previously introduced relaxation systems in terms of isotropic coverage of the multi-dimensional domain by the foot of the characteristic, a finite volume method is developed in which the fluxes at the cell interfaces are evaluated in a genuinely multi-dimensional way, in contrast to the traditional dimension-by-dimension treatment. This algorithm is tested on some bench-mark test problems for hyperbolic conservation laws.

**Keywords :** genuinely multi-dimensional schemes, relaxation systems, isotropy, hyperbolic conservation laws

## 1 Introduction

Finite volume methods have been popular for the numerical solution of hyperbolic conservation laws in the last three decades. For multi-dimensional flows, however, the traditional finite volume methods are typically based on a dimension-by-dimension treatment using upwind discretizations. As a result of this inherently one-dimensional treatment, the discontinuities which are oblique to the coordinate directions are not resolved well. Developing genuinely multi-dimensional algorithms has been a topic of intense research in the last decade and a half. The reader is referred to [1], [2], [3] for some of the developments in this topic. Of all the upwind schemes developed in the past decades, the relaxation schemes are the simplest, as they do not involve the solution of any Riemann problems or complicated flux splitting and are just based on linear convection equations. Aregba-Driollet and Natalini [4] have introduced a multidimensional relaxation system in the form of a discrete velocity BGK Boltzmann equation. In their model, the foot of the bicharacteristic curves through any point does not fall evenly around the point. To overcome this deficiency Manisha *et.al* [5] have given an isotropic relaxation system which incorporates the multidimensional nature in a more realistic way. We have developed a genuinely multidimensional finite volume scheme based on the relaxation system in [5], following the work of Lukáčová *et.al* [3].

## 2 Relaxation systems

In this section we give our relaxation system. For the sake of simplicity we present the details for a single conservation law in two dimensions. The extension to systems is straight forward. Consider the conservation law,

$$\partial_t u + \partial_x g_1(u) + \partial_y g_2(u) = 0 \quad (2.1)$$

The relaxation system for (2.1) is given by

$$\partial_t \mathbf{f} + \Lambda_1 \partial_x \mathbf{f} + \Lambda_2 \partial_y \mathbf{f} = \frac{1}{\epsilon} (\mathbf{F} - \mathbf{f}) \quad (2.2)$$

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where,  $\mathbf{f} = (f_1, f_2, f_3, f_4)^t$ ,  $\mathbf{F} = (F_1, F_2, F_3, F_4)^t$  and the matrices  $\Lambda_1$  and  $\Lambda_2$  are

$$\Lambda_1 = \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} \quad \text{and} \quad \Lambda_2 = \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}. \quad (2.3)$$

$F$  is called the local Maxwellian and its components are defined by

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} \frac{u}{4} - \frac{g_1(u)}{4\lambda} - \frac{g_2(u)}{4\lambda} \\ \frac{u}{4} + \frac{g_1(u)}{4\lambda} - \frac{g_2(u)}{4\lambda} \\ \frac{u}{4} + \frac{g_1(u)}{4\lambda} + \frac{g_2(u)}{4\lambda} \\ \frac{u}{4} - \frac{g_1(u)}{4\lambda} + \frac{g_2(u)}{4\lambda} \end{bmatrix} \quad (2.4)$$

The system (2.2) has four families of bicharacteristics and the feet of the bicharacteristics through any point  $(x, y, t + \Delta t)$  falls in all the four quadrants around the point  $(x, y, t)$ . See Figure(1).

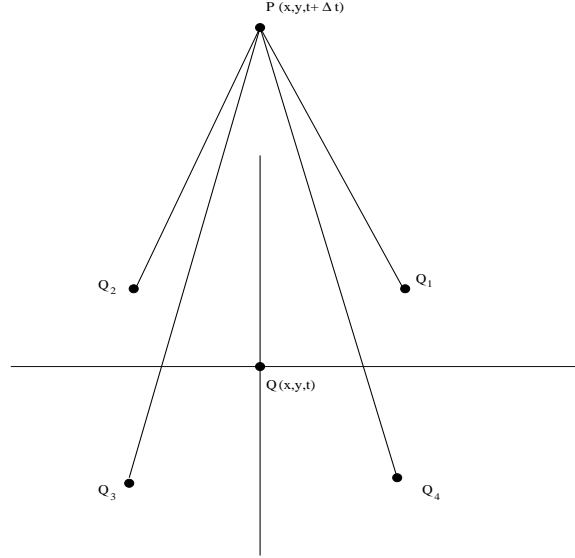


Figure 1: Feet of the bicharacteristics

The parameter  $\lambda$  in the above relaxation system is chosen according to the stability condition given by a Chapman-Enskog type expansion. See [4] for details. In the present case the above analysis yields,

$$\lambda^2 \geq (\partial_u g_1(u))^2 + (\partial_u g_2(u))^2 \quad (2.5)$$

For systems, the form of the relaxation system (2.2) is the same, but now  $f_1, f_2$  etc are themselves vectors.

### 3 A numerical scheme based on bicharacteristics

The discrete Boltzmann equation (2.2) is solved by splitting method as,

$$\partial_t \mathbf{f} + \Lambda_1 \partial_x \mathbf{f} + \Lambda_2 \partial_y \mathbf{f} = 0 \quad (\text{convection step}) \quad (3.6)$$

$$\frac{d\mathbf{f}}{dt} = \frac{1}{\epsilon} (\mathbf{F} - \mathbf{f}) \quad (\text{relaxation step}). \quad (3.7)$$

The relaxation step is further simplified by taking  $\epsilon = 0$ , leading to  $\mathbf{f} = \mathbf{F}$ . So at any stage we need to solve only the set of linear convection equations in (3.6). If we integrate (3.6) over a mesh cell and over the time interval from  $n\Delta t$  to  $(n+1)\Delta t$ , application of Gauss formula gives,

$$\mathbf{f}^{n+1} = \mathbf{f}^n - \Delta t \int_0^{\Delta t} \left[ \Lambda_1 \delta_x \mathbf{f}^{n+\tau/\Delta t} + \Lambda_2 \delta_y \mathbf{f}^{n+\tau/\Delta t} \right] d\tau. \quad (3.8)$$

In this formula,  $\mathbf{f}^{n+1}$  and  $\mathbf{f}^n$  represents the cell averages, while  $\delta_x \mathbf{f}^{n+\tau/\Delta t}$  involves averages along the cell edges to the right and left and  $\delta_y \mathbf{f}^{n+\tau/\Delta t}$  along the edges to the top and bottom, in all cases at an intermediate time step  $n\Delta t + \tau$ . We approximate the time integrals in (3.8) by midpoint rule by  $\tau = \Delta t/2$ . Then the cell boundary flux  $\mathbf{f}^{n+1/2}$  is evolved using the exact solution  $E_{\Delta t/2}$  of (3.6) and suitable recovery operators  $R_h$ . For example on vertical edges,

$$\mathbf{f}^{n+1/2} = \frac{1}{h} \int_0^h E_{\Delta t/2} R_h \mathbf{f}^n ds_y, \quad (3.9)$$

where  $ds_y$  is an element of arc length along the vertical edges. A similar expression holds for horizontal edges also. We have used the Simpson's rule for the cell interface integrals in (3.9) and evaluated the fluxes at the midpoints of the edges and corners. Note that in this way Simpson's quadrature takes into account of the multidimensional effects.

## 4 Numerical results

*2-D Burgers' equation test case* : This test case is taken from Spekreijse [6]. It models a shock and an expansion fan in a 2-D domain. The results are given in Figure(2).

*2-D Euler equations test case* : This problem is taken from Lukáčová *et.al* [3]. The problem contains a circular shock and circular contact discontinuity moving away from the centre of the circle and circular rarefaction wave moving in the opposite direction. The results are given in Figure(3).

The presented results for the Euler equations are only of first order accuracy. Higher order results and the comparison with schemes based on dimension-by-dimension upwinding will be given later. As can be seen from the results, all the features of the solution are resolved very well, even with first order accurate version of the scheme.

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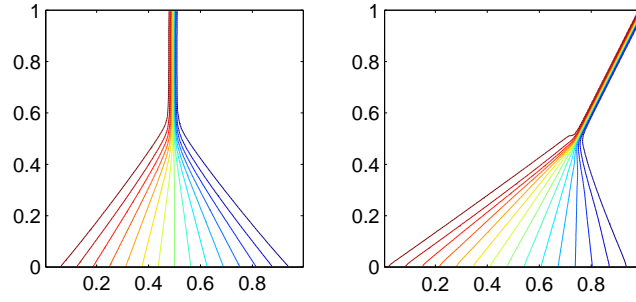


Figure 2: Burgers' equation test cases

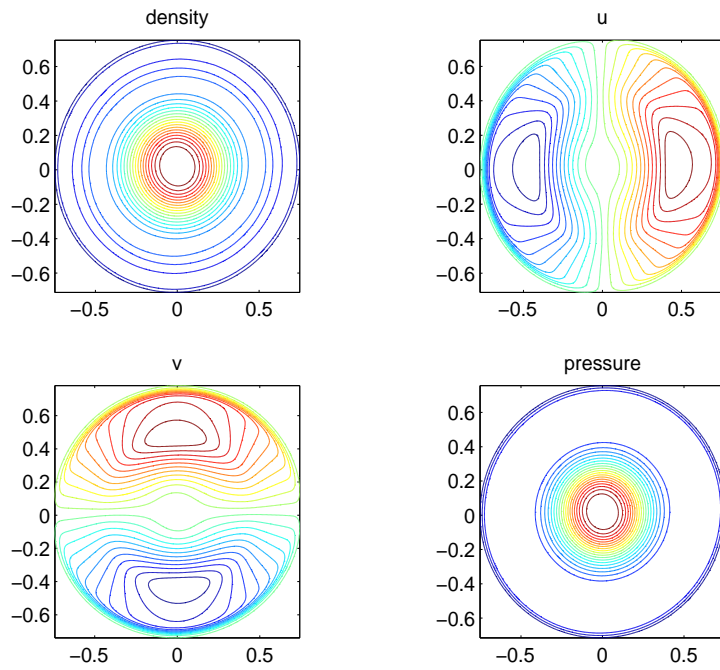


Figure 3: Euler equations test case