INTERMEDIATE REDUCTION STEPS IMPROVE AUTOMATED MULTI-LEVEL SUBSTRUCTURING

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Abstract

The Automated Multi-Level Substructuring (AMLS) method has been developed to reduce the computational demands of frequency response analysis and has recently been proposed as an alternative to iterative projection methods like Lanczos or Jacobi–Davidson for computing a large number of eigenvalues for matrices of very large dimension. Based on Schur complements and modal approximations of submatrices on several levels AMLS constructs a projected eigenproblem which yields good approximations of eigenvalues at the lower end of the spectrum. In this paper we reduce the cost of AMLS introducing intermediate reduction steps. The efficiency of this approach is demonstrated by a huge gyroscopic eigenvalue problem modelling the dynamic behaviour of a rotating tyre.

Keywords: automated multi-level substructuring, AMLS, intermediate reduction, eigenproblem, eigenvalue, eigenvector, sparse matrix, iterative projection method, Arnoldi method, nonlinear eigenvalue problem

1 Introduction

Over the last few years, a new method for huge linear eigenvalue problems

$$Kx = \lambda Mx \tag{1}$$

where $K \in \mathbb{C}^{n\times n}$ and $M \in \mathbb{C}^{n\times n}$ are Hermitian and positive definite, known as *Automated Multi–Level Substructuring (AMLS)*, has been developed by Bennighof and co-authors, and has been applied to frequency response analysis of complex structures [1, 2, 3, 4]. Here the large finite element model is recursively divided into very many substructures on several levels based on the sparsity structure of the system matrices. Assuming that the interior degrees of freedom of substructures depend quasistatically on the interface degrees of freedom, and modelling the deviation from quasistatic dependence in terms of a small number of selected substructure eigenmodes the size

of the finite element model is reduced substantially yet yielding satisfactory accuracy over a wide frequency range of interest. Recent studies ([12, 14, 20], e.g.) in vibro-acoustic analysis of passenger car bodies, where very large FE models with more than six million degrees of freedom appear and several hundreds of eigenfrequencies and eigenmodes are needed, have shown that for this type of problems AMLS is considerably faster than Lanczos type approaches.

On each level of the hierarchical substructuring AMLS consists of two steps. First for every substructure of the current level a congruence transformation based on block-Gaussian elimination is applied to the matrix pencil to decouple in the stiffness matrix the substructure from the degrees of freedom of higher levels. Secondly, the dimension of the problem is reduced by modal truncation of the corresponding diagonal blocks discarding eigenmodes according to eigenfrequencies which exceed a predetermined cut-off frequency. Hence, AMLS is nothing else but a projection method where the large problem under consideration is projected to a subspace spanned by a smaller number of eigenmodes of undamped clamped substructures on several levels. With respect to this basis the projection of the stiffness matrix K becomes diagonal, and the mass matrix M is projected to a generalised arrowhead form.

The a priori error bound of AMLS in [8] overestimates the true error by orders of magnitude, and realistic estimates of the dependence of the relative errors of eigenvalues on the cut-off frequency are not known, but it is common experience that for all eigenvalues not exceeding 10% of the cut-off frequency the relative error is less than 1%. Hence, if a very large frequency range is needed then AMLS has to be applied with a high cut-off frequency resulting in a projected problem of still quite large dimension.

In this presentation we discuss a way how to reduce the dimension of the projected problem on-the-fly by intermediate reduction steps applying modal condensation to aggregated substructures which consist of the union of already reduced substructures corresponding to a subtree of the partitioning graph.

The paper is organised as follows. Section 2 summarises the automated multi-level substructuring method for linear eigenvalue problems. Section 3 discusses its improvement by intermediate reduction steps, and Section 4 demonstrates the improvement of AMLS by intermediate reduction steps for a huge gyroscopic eigenvalue problem modeling the dynamic behaviour of rotating tyre.

2 Automated Multi-Level Substructuring

In this section we summarise the *Automated Multi-Level Substructuring* (AMLS) method for the linear eigenvalue problem

$$Kx = \lambda Mx \tag{2}$$

which was developed by Bennighof and co-workers over the last few years [2, 4, 12], who applied it to solve frequency response problems involving large and complex

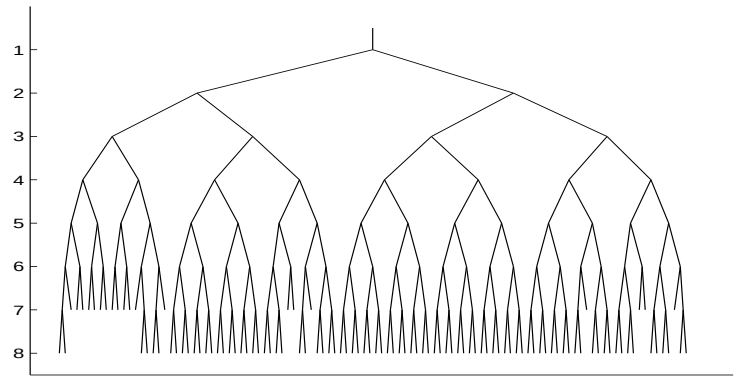

Fig. 1: Substructure tree

models. Here, K is the stiffness matrix and M the mass matrix of a finite element model of a structure. Both matrices are assumed to be large and sparse, and are symmetric and positive definite.

Similarly as in the component mode synthesis method (CMS) the structure is partitioned into a small number of substructures based on the sparsity pattern of the system matrices, but more generally than in CMS these substructures in turn are substructured on a number of levels yielding a tree topology for the substructures. Compare Figure 1 where the nodes correspond to substructures, and subtrees subordinate to a node indicate its further offspring.

We stress the fact that substructuring does not mean that it is obtained by a domain decomposition of a real structure, but it is understood in a purely algebraic sense, i.e. the dissection of the matrices can be derived by applying a graph partitioner like CHACO [10] or METIS [13] to the matrix under consideration. However, because of its pictographic nomenclature we will use terms like substructure or eigenmode from frequency response problems when describing the AMLS method.

Substructures on the lowest level consist of a small number of degrees of freedom, which are partitioned into two sets: interface degrees of freedom which are shared with an adjacent substructure, and interior or local degrees of freedom which are only connected to degrees of freedom in their own substructure. After reordering problem (2) obtains the following form

$$\begin{pmatrix} K_{\ell\ell}^{(1)} & K_{\ell i}^{(1)} \\ K_{i\ell}^{(1)} & K_{ii}^{(1)} \end{pmatrix} \begin{pmatrix} x_{\ell}^{(1)} \\ x_{i}^{(1)} \end{pmatrix} = \lambda \begin{pmatrix} M_{\ell\ell}^{(1)} & M_{\ell i}^{(1)} \\ M_{i\ell}^{(1)} & M_{ii}^{(1)} \end{pmatrix} \begin{pmatrix} x_{\ell}^{(1)} \\ x_{i}^{(1)} \end{pmatrix} \tag{3}$$

where $x_{\ell}^{(1)}$ and $x_{i}^{(1)}$ denote the local and interface degrees of freedom, respectively,

and due to the substructuring on the finest level

$$K_{\ell\ell}^{(1)} = \begin{pmatrix} K_{\ell\ell 1}^{(1)} & & & \\ & K_{\ell\ell 2}^{(1)} & & \\ & & \ddots & \\ & & & K_{\ell\ell r_1}^{(1)} \end{pmatrix}$$

is a block diagonal matrix and $M_{\ell\ell}^{(1)}$ has the same structure.

We first decouple the local degrees of freedom from the interface ones in the stiffness matrix. To this end we apply the variable transformation

$$\begin{pmatrix} x_\ell^{(1)} \\ x_i^{(1)} \end{pmatrix} = \begin{pmatrix} I & -(K_{\ell\ell}^{(1)})^{-1} K_{\ell i}^{(1)} \\ O & I \end{pmatrix} \begin{pmatrix} \tilde{x}_\ell^{(1)} \\ x_i^{(1)} \end{pmatrix} =: U\tilde{x}^{(1)}, \tag{4}$$

and we multiply by U^H from the left to retain the symmetry of the eigenvalue problem. This yields the equivalent eigenvalue problem

$$\begin{pmatrix} K_{\ell\ell}^{(1)} & \\ & \tilde{K}_{ii}^{(1)} \end{pmatrix} \begin{pmatrix} \tilde{x}_\ell^{(1)} \\ x_i^{(1)} \end{pmatrix} = \lambda \begin{pmatrix} M_{\ell\ell}^{(1)} & \tilde{M}_{\ell i}^{(1)} \\ \tilde{M}_{i\ell}^{(1)} & \tilde{M}_{ii}^{(1)} \end{pmatrix} \begin{pmatrix} \tilde{x}_\ell^{(1)} \\ x_i^{(1)} \end{pmatrix}. \tag{5}$$

Here $K_{\ell\ell}^{(1)}$ and $M_{\ell\ell}^{(1)}$ stay unchanged, and

$$\begin{aligned} \tilde{K}_{ii}^{(1)} &= K_{ii}^{(1)} - K_{i\ell}^{(1)}(K_{\ell\ell}^{(1)})^{-1}K_{\ell i}^{(1)} \quad \text{is the Schur complement of } K_{\ell\ell}^{(1)} \\ \tilde{M}_{\ell i}^{(1)} &= M_{\ell i}^{(1)} - M_{\ell\ell}^{(1)}(K_{\ell\ell}^{(1)})^{-1}K_{\ell i}^{(1)} = (\tilde{M}_{i\ell}^{(1)})^H \\ \tilde{M}_{ii}^{(1)} &= M_{ii}^{(1)} - M_{i\ell}^{(1)}(K_{\ell\ell}^{(1)})^{-1}K_{\ell i}^{(1)} - K_{i\ell}^{(1)}(K_{\ell\ell}^{(1)})^{-1}M_{\ell i}^{(1)} \\ &\quad + K_{i\ell}^{(1)}(K_{\ell\ell}^{(1)})^{-1}M_{\ell\ell}^{(1)}(K_{\ell\ell}^{(1)})^{-1}K_{\ell i}^{(1)}. \end{aligned}$$

Notice, that $K_{\ell\ell}^{(1)}$ is block diagonal, where each diagonal block corresponds to a substructure on the finest level. Since the part of a column of $K_{\ell i}^{(1)}$ which belongs to the jth diagonal block of $K_{\ell\ell}^{(1)}$ is only different from the null vector, if the corresponding degree of freedom is located on the boundary of the jth substructure, $(K_{\ell\ell}^{(1)})^{-1}K_{\ell i}^{(1)}$ can be determined very efficiently (and in parallel).

Next we reduce the dimension of the problem by modal reduction of the finest level. To this end we solve the partial eigenvalue problem

$$K_{\ell\ell}^{(1)}\phi_j = \omega_j M_{\ell\ell}^{(1)}\phi_j, \quad \phi_i^H M_{\ell\ell}^{(1)}\phi_j = \delta_{ij}, \tag{6}$$

and we project problem (5) to the space spanned by the columns of $P = \mathrm{diag}[P_1, I_i]$, where $\dim I_i = \dim K_{ii}^{(1)}$, and where the columns of P_1 are the eigenmodes of problem (6) corresponding to eigenvalues which do not exceed a given cut-off frequency γ.

Again due to the block structure of $K_{\ell\ell}^{(1)}$ and $M_{\ell\ell}^{(1)}$ problem (6) decouples into r_1 eigenproblems of small dimension which can be solved independently.

We arrive at a reduced eigenvalue problem

$$\begin{pmatrix} \Omega_1 & \\ & \tilde{K}_{ii}^{(1)} \end{pmatrix} \begin{pmatrix} \hat{x}_\ell^{(1)} \\ x_i^{(1)} \end{pmatrix} = \lambda \begin{pmatrix} I & \hat{M}_{\ell i}^{(1)} \\ \hat{M}_{i\ell}^{(1)} & \tilde{M}_{ii}^{(1)} \end{pmatrix} \begin{pmatrix} \hat{x}_\ell^{(1)} \\ x_i^{(1)} \end{pmatrix} \tag{7}$$

where Ω_1 is a diagonal matrix containing the retained eigenvalues ω_j in its diagonal, $\hat{x}_\ell^{(1)} = P_1 \tilde{x}_\ell^{(1)}$ and $\hat{M}_{\ell i}^{(1)} = P_1^H \tilde{M}_{\ell i}^{(1)}$.

(7) is the eigenvalue problem which results from the component mode synthesis method (CMS method for short) introduced by Craig and Bampton [5].

Once substructures on the finest level have been transformed and reduced by modal projection they are assembled to parent substructures. Reordering the unknowns again the eigenvalue problem obtains the following form

$$\begin{pmatrix} K_{\ell\ell}^{(2)} & K_{\ell i}^{(2)} \\ K_{i\ell}^{(2)} & K_{ii}^{(2)} \end{pmatrix} \begin{pmatrix} x_\ell^{(2)} \\ x_i^{(2)} \end{pmatrix} = \lambda \begin{pmatrix} M_{\ell\ell}^{(2)} & M_{\ell i}^{(2)} \\ M_{i\ell}^{(2)} & M_{ii}^{(2)} \end{pmatrix} \begin{pmatrix} x_\ell^{(2)} \\ x_i^{(2)} \end{pmatrix}. \tag{8}$$

The matrices

$$K_{\ell\ell}^{(2)} = \mathrm{diag}[K_{\ell\ell 1}^{(2)}, \ldots, K_{\ell\ell r_2}^{(2)}] \quad \text{and} \quad M_{\ell\ell}^{(2)} = \mathrm{diag}[M_{\ell\ell 1}^{(2)}, \ldots, M_{\ell\ell r_2}^{(2)}]$$

are block diagonal with

$$K_{\ell\ell j}^{(2)} = \begin{pmatrix} \Omega_{j1}^{(1)} & & & \\ & \ddots & & \\ & & \Omega_{j\nu_j}^{(1)} & \\ & & & \tilde{K}_{\ell\ell j}^{(2)} \end{pmatrix}, \; M_{\ell\ell j}^{(2)} = \begin{pmatrix} I & & & M_{j1}^{(2)} \\ & \ddots & & \vdots \\ & & I & M_{j\nu_j}^{(2)} \\ (M_{j1}^{(2)})^T & \ldots & (M_{j\nu_j}^{(2)})^T & \tilde{M}_{\ell\ell j}^{(2)} \end{pmatrix}. \tag{9}$$

ν_j denotes the number of children which are assembled to the jth parent substructure, and for $i = 1, \ldots, \nu_j$ the ith block row and column of $K_{\ell\ell j}^{(2)}$ and $M_{\ell\ell j}^{(2)}$ correspond to the ith substructure of the finest level. The entries in $K_{\ell i}^{(2)}$ corresponding to these blocks are zero, the remaining ones describe the coupling of the jth substructure to interface degrees on coarser levels.

Similar to the finest level we use block Gaussian elimination to decouple the local degrees of freedom $x_\ell^{(2)}$ in the stiffness matrix from the interface unknowns $x_i^{(2)}$. Continuing as on the finest level we would reduce the dimension of the local degrees of freedom by modal reduction solving the eigenvalue problems

$$K_{\ell\ell j}^{(2)} \phi_j = \omega_j M_{\ell\ell j}^{(2)} \phi_j, \; j = 1, \ldots, r_2,$$

where the matrices are given in (9).

However, since the number of interior degrees of freedom of substructures grows too large in the course of the algorithm, in standard AMLS one does not reduce the local degrees of freedom of the jth structure of the current level by modal reduction

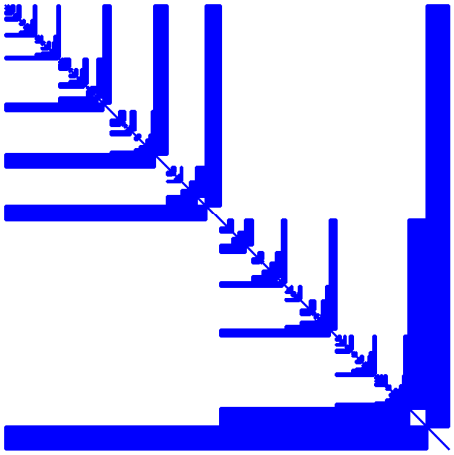

Fig. 2: Projected mass matrix

but only the interface degrees of freedom of the next finer level which are contained in the jth substructure, i.e. one employs only the eigenvalue problem corresponding to the right lower diagonal block

$$\tilde{K}^{(2)}_{\ell\ell j}\phi_j = \omega_j \tilde{M}^{(2)}_{\ell\ell j}\phi_j$$

in order to reduce the j substructure.

Assembly to higher-level substructures, transformation to quasistatic-modal representation, and projection to subspaces spanned by modes of diagonal blocks not exceeding the cut-off frequency continues, until we arrive at an approximate model for the entire structure. We end up with a projected eigenproblem

$$\mathcal{K}y = \lambda \mathcal{M}y, \tag{10}$$

where the condensed matrices $\mathcal{K}$ and $\mathcal{M}$ are symmetric and positive definite. The stiffness matrix $\mathcal{K}$ has become diagonal containing the retained eigenvalues of substructures in its diagonal, and the mass matrix is projected to a matrix $\mathcal{M}$ the diagonal of which is the identity, and the only off-diagonal blocks containing non-zero elements are the ones describing the coupling of the substructures and its interfaces. Figure 2 shows the generalised arrowhead structure of a resulting mass matrix.

3 Intermediate reduction

Dynamic analysis of structures frequently involves finite element discretisations with millions of unknowns, and the frequency range needed for the analysis often is so large that the number of required eigenpairs can easily reach into the thousands. In this situation it may happen that the dimension of the reduced eigenproblem (10) is still so high that the CPU time required for solving it is much larger than the CPU time for reducing the original problem to its condensed form (10).

We already pointed out that in standard AMLS one applies modal reduction only to the part of a substructure that corresponds to the unknowns which are interface degrees of freedom on the next finer level but not to the entire substructure (including the offspring of the current substructure). This keeps the time for modal truncation of substructures on coarser levels small, but results in a quite large projected eigenvalue problem (10).

An idea at hand is to balance the CPU time for performing the reduction and the CPU time for solving the final projected problem introducing intermediate modal truncations.

To be more specific, consider a fixed substructure on some intermediate level, and assume that Gaussian elimination has been already applied to decouple it in the stiffness matrix from its interface degrees of freedom. Then the unknowns can be ordered such that the current problem has the form

$$\begin{pmatrix} k_s & & \\ & K_m & \\ & & K_r \end{pmatrix} \begin{pmatrix} x_s \\ x_m \\ x_r \end{pmatrix} = \lambda \begin{pmatrix} m_s & m_{sm} & m_{sr} \\ m_{ms} & M_m & M_{mr} \\ m_{rs} & M_{rm} & M_r \end{pmatrix} \begin{pmatrix} x_s \\ x_m \\ x_r \end{pmatrix} \tag{11}$$

where x_s are the coordinates retained in the modal reduction of its substructures on previous levels, x_m are the newly included unknowns (which are interface degrees of freedom on the next finer level), and x_r are the remaining degrees of freedom corresponding to other substructures on the same level and to ancestors in the substructure tree.

Modal reduction with respect to x_m yields

$$\begin{pmatrix} k_s & & \\ & k_m & \\ & & K_r \end{pmatrix} \begin{pmatrix} x_s \\ \tilde{x}_m \\ x_r \end{pmatrix} = \lambda \begin{pmatrix} m_s & m_{sm} & m_{sr} \\ m_{ms} & I & m_{mr} \\ m_{rs} & m_{rm} & M_r \end{pmatrix} \begin{pmatrix} x_s \\ \tilde{x}_m \\ x_r \end{pmatrix} \tag{12}$$

and standard AMLS would proceed to the next coarser level. To keep the dimension of the final condensed problem at a reasonable size we apply a second modal reduction if the substructure, i.e. the dimension of $\begin{pmatrix} x_s \\ \tilde{x}_m \end{pmatrix}$ has become too large, this time taking into account the retained children degrees of freedom x_s as well. Thus, the intermediate modal reduction is applied to the system

$$\begin{pmatrix} k_s & \\ & k_m \end{pmatrix} \begin{pmatrix} x_s \\ \tilde{x}_m \end{pmatrix} = \omega \begin{pmatrix} m_s & m_{sm} \\ m_{ms} & I \end{pmatrix} \begin{pmatrix} x_s \\ \tilde{x}_m \end{pmatrix} \tag{13}$$

with the same cut-off frequency as for the standard reduction steps. Notice, that the mass matrix of (13) has the same generalised arrow head structure as the final condensed matrix, but its dimension is smaller, and - more important - it keeps the dimension of the final projected problem smaller, although more populated.

4 Numerical experiments

To evaluate AMLS with intermediate reduction we consider the gyroscopic eigenvalue problem

$$Q(\omega)x := Kx + i\omega Gx - \omega^2 Mx = 0 \tag{14}$$

governing eigenvibrations of rotating structures. Here K is the stiffness matrix modified by the presence of centripetal forces, M is the mass matrix, and G is the gyroscopic matrix stemming from the Coriolis force. Clearly, K and M are symmetric and positive definite, and G is skew–symmetric.

For example, this problem arises when modeling noise of rolling tyres which is the major source of traffic noise for passenger cars at speeds exceeding 40 km/h. Due to the complicated interior structure of a belted tire the matrices K, M and G of a sufficiently accurate FE model are very large and sparse. Moreover, for the acoustic analysis many eigenpairs not necessarily at the end of the spectrum are needed. Therefore, well-established sparse eigensolvers of Arnoldi type with shift and invert techniques [15] for a linearisation of problem (14), methods which are based on structure preserving linearisations like SHIRA [17], and iterative projection methods for nonlinear eigenproblems [16] are very costly since LU factorisations of complex valued matrices $Q(\omega_j)$ for several parameters ω_j are required.

In particular we consider a tyre model with 39204 brick elements, 124992 degrees of freedom and 20 different material groups, rotating with 50 km/h. Our aim is to determine approximations to the smallest 200 eigenvalues with relative error less than 0.2% and the corresponding eigenvectors.

Linearising in the usual way

$$\begin{pmatrix} -iG & -K \\ I & O \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix} = \omega \begin{pmatrix} M & O \\ O & I \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix} \tag{15}$$

or by the Hermitian problem

$$\begin{pmatrix} iG & K \\ K & O \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix} = \omega \begin{pmatrix} M & O \\ O & K \end{pmatrix} \begin{pmatrix} \omega x \\ x \end{pmatrix} \tag{16}$$

and applying the shift-and-invert Arnoldi method requires an LU factorisation of $Q(\omega) = K + i\omega G - \omega^2 M$ for every shift ω, which is a complex matrix.

Determining the factorisation by SuperLU requires a memory of 6.04 GByte and a CPU time of 987 seconds on an SGI Altix 4700, Itanium2 Madison 9M processor (1.6 GHz / 6MB L3 Cache), by the more efficient parallel direct solver PARDISO [18, 19] 2.23 GByte and 115 seconds.

Applying the nonlinear Arnoldi method [21] the preconditioners can be chosen as real matrices $K - \omega^2 M$, the LU factorisation of which requires 2.7 GByte storage and 485 seconds with SuperLU [6], 2.86 GByte storage and 267 seconds with the multi frontal solver MA57 of HSL [11], and 1.11 GByte storage and 70 seconds with PARDISO. Since the LU factorisation has to be updated several times a total CPU time of more than 4.66 hours with MA57 results, and 3 hours with PARDISO.

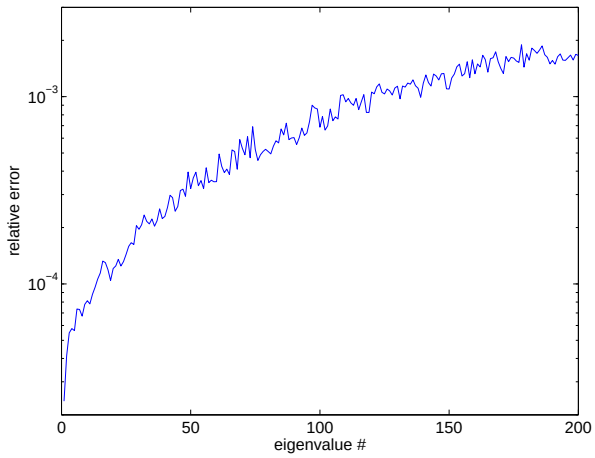

Fig. 3: Relative errors of smallest 200 eigenvalues

AMLS could be applied to the Hermitian linear eigenproblem (16). However, since the influence of the gyroscopic matrix G on the eigenvalues is usually not very high compared to the mass and stiffness matrix, we construct the projected problem neglecting the term $i\omega G$, applying AMLS to the linear eigenproblem $Kx = \omega^2 Mx$, and transforming and projecting the matrix G simultaneously (cf. [7, 9]).

Since the sparsity pattern of G matches the ones of K and M one gets the reduced model

$$\mathcal{K}y + i\omega\mathcal{G}y - \omega^2\mathcal{M}y = 0, \tag{17}$$

where the stiffness and mass matrix have the same structure as in the linear case, and the gyroscopic matrix $\mathcal{G}$ is a skew-symmetric block matrix containing diagonal blocks corresponding to the (reduced) substructures and interfaces, and only off–diagonal blocks describing the coupling of a substructure and its interface contain non–zero elements. Notice, that all projectors are real, and therefore the reduction can be performed in real arithmetic.

AMLS demands only 2.0 GByte storage and the problem under consideration can be solved on a personal computer, namely a 2.2 GHz Intel Xeon processor.

With a cut-off frequency of $\omega_c = 1.45 \times 10^{10}$ the problem is projected to a gyroscopic eigenproblem of dimension $n_c = 5571$ requiring a CPU time of 1021 seconds. Solving the linearisation (16) of the projected problem (17) under MATLAB 7.0 by `eigs` (i.e. by ARPACK) requires another 2150 seconds, i.e. a total computing time of 3171 seconds.

Figure 3 shows the relative errors of all 200 eigenvalues which are all smaller than 0.19%.

Introducing intermediate reduction steps whenever the dimension of a reduced model of a substructure exceeds 800 requires 1184 seconds for the reduction and 889 for solving the reduced eigenproblem, i.e. a total computing time of 2073 seconds. However, the maximum relative error increases to 0.23%.

To meet the accuracy requirement we have to increase the cut-off frequency to

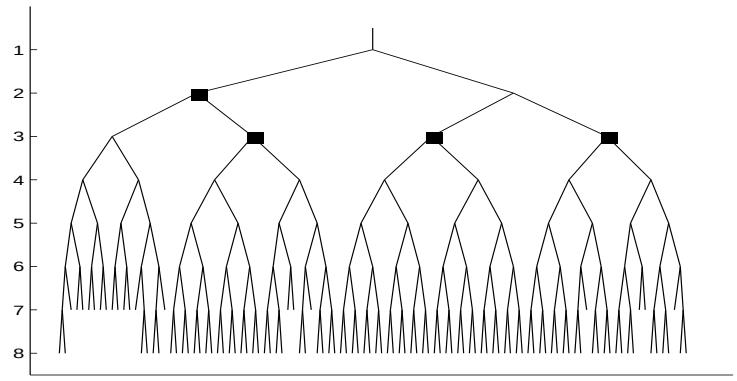

Fig. 4: Substructure tree; intermediate reduction

$\omega_c = 2.1 \times 10^{10}$. Then the maximum relative error is again 0.19% (a semilogarithmic plot of the relative error looks very similar to the one in Figure 3 without intermediate reduction), the dimension of the reduced problem is 1922, the CPU time for reducing the problem is 1541 seconds, and the CPU time for solving the reduced problem is 925 seconds. Hence, the total CPU time is reduced to 2466 seconds. Four additional intermediate reductions are applied which are tagged in the substructure tree in Figure 4.

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