2D Navier-Stokes Equations in a Time Dependent Domain with Neumann Type Boundary Conditions

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Abstract. In this paper the two-dimensional Navier-Stokes system for incompressible fluid coupled with a parabolic equation through the Neumann type boundary condition for the second component of the velocity is considered. Navier-Stokes equations are defined on a given time dependent domain. We prove the existence of a weak solution for this system. In addition, we prove the continuous dependence of solutions on the data for a regularized version of this system. For a special case of this regularized system also a problem with an unknown interface is solved.

The problem under consideration is an approximation of the fluid-structure interaction problem proposed by A. Quarteroni in [19]. We conjecture that our approach is useful also for the numerical treatment of the problem and at the end we shortly present our numerical experiments.

Mathematics Subject Classification. 35Q30, 65M20, 74F10, 76D03
Keywords. Navier-Stokes equations, Time dependent domain, Fluid-structure interaction, Neumann boundary condition

1. Introduction

We consider the two-dimensional Navier–Stokes system for incompressible fluid

$$\rho \frac{\partial v_i}{\partial t} + \rho \sum_{j=1}^{2} v_j \frac{\partial v_i}{\partial x_j} = \mu \Delta v_i - \frac{\partial p}{\partial x_i} \qquad i = 1, 2 \,, \qquad \operatorname{div} \boldsymbol{v} \, = 0 \tag{1.1}$$

in a domain

$$\Omega(h) \equiv \{(x_1, x_2, t) : \; 0 < x_1 < L, \; 0 < x_2 < h(x_1, t), \; 0 < t < T\}$$

for a given C^1 function h satisfying

$$0 < \alpha \le h(x_1, t) \le \alpha^{-1} \,, \qquad h(x_1, 0) = h(0, t) = h(L, t) = R > 0 \,. \tag{1.2}$$

On the upper part of the boundary of $\Omega(h)$, that we shall denote $\Gamma_w = \Gamma_{wall}$, we require $v_1 = 0$, i.e.

$$v_1(x_1, h(x_1, t), t) = 0 \qquad \text{for} \quad 0 < x_1 < L, \; 0 < t < T \tag{1.3}$$

*Supported by the Grants of the Comenius University 66/2004 and 321/2005
Both authors were supported by the VEGA Grant Agency of the Slovak Republic No. 1/0260/03

and we impose the following Neumann type boundary condition for the second component of the velocity $\boldsymbol{v}$

$$\left[\mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial h}{\partial x_1}\right)+\mu\frac{\partial v_2}{\partial x_2}-p+p_W-\frac{\rho}{2}v_2\left(v_2-\frac{\partial h}{\partial t}\right)\right](x_1,h(x_1,t),t)$$
$$=\rho\kappa\left(\lambda\frac{\partial\eta}{\partial t}(x_1,t)+(1-\lambda)\frac{\partial h}{\partial t}(x_1,t)-v_2(x_1,h(x_1,t),t)\right) \tag{1.4}$$

for a given function $p_w = p_w(x_1,t)$, $0<\lambda\le 1$ and some $\kappa \gg 1$. In this boundary condition an unknown function $\eta=\eta(x_1,t)$ appears. We require from η to satisfy the following differential equation

$$-E\left[\frac{\partial^2\eta}{\partial t^2}-a\frac{\partial^2\eta}{\partial x_1^2}+b\eta-c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right](x_1,t) \tag{1.5}$$
$$=\kappa\left(\lambda\frac{\partial\eta}{\partial t}(x_1,t)+(1-\lambda)\frac{\partial h}{\partial t}(x_1,t)-v_2(x_1,h(x_1,t),t)\right)$$

for any $0<x_1<L$, $0<t<T$ equipped with the boundary and initial condition for η,

$$\eta(0,t)=\eta(L,t)=0 \qquad \text{and} \qquad \eta(x_1,0)=\frac{\partial\eta}{\partial t}(x_1,0)=0\ . \tag{1.6}$$

Here $\rho,\mu,L,T,\alpha,R,E,a,b,c$ are given positive constants.

According to our motivation, described in Section 2 below, we complete the Navier–Stokes system (1.1) with the following boundary and initial conditions. On a part of the boundary which we shall denote $\Gamma_{in}=\Gamma_{inflow}$ we put

$$v_2(0,x_2,t)=0\ , \qquad \left(\mu\frac{\partial v_1}{\partial x_1}-p+p_{in}-\frac{\rho}{2}|\boldsymbol{v}|^2\right)(0,x_2,t)=0 \tag{1.7}$$

for any $0<x_2<R$, $0<t<T$ and for a given function $p_{in}=p_{in}(x_2,t)$. On the opposite part of the boundary $\Gamma_{out}=\Gamma_{outflow}$ we put

$$v_2(L,x_2,t)=0\ , \qquad \left(\mu\frac{\partial v_1}{\partial x_1}-p+p_{out}-\frac{\rho}{2}|\boldsymbol{v}|^2\right)(L,x_2,t)=0 \tag{1.8}$$

for any $0<x_2<R$, $0<t<T$ and for a given function $p_{out}=p_{out}(x_2,t)$. Finally, on the rest part of the boundary Γ_c we set

$$v_2(x_1,0,t)=0\ , \qquad \mu\frac{\partial v_1}{\partial x_2}(x_1,0,t)=0 \tag{1.9}$$

for any $0<x_1<L$, $0<t<T$ and

$$\boldsymbol{v}(x_1,x_2,0)=\mathbf{0} \qquad \text{for any } 0<x_1<L,\ 0<x_2<h(x_1,0)\ . \tag{1.10}$$

Problem (1.1)–(1.10) is an approximation of the fluid–structure interaction model proposed by A. Quarteroni in [19]. In the original proposal h is not known, instead, $\boldsymbol{v}$, p and a free boundary

$$h=R+\eta \tag{1.11}$$

are to be found such that (1.4) and (1.5) are replaced by

$$\mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial \eta}{\partial x_1}\right)+\mu\frac{\partial v_2}{\partial x_2}-p+p_w \;=\; -E\rho\left[\frac{\partial^2\eta}{\partial t^2}-a\frac{\partial^2\eta}{\partial x_1^2}+b\eta-c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right] \quad (1.12)$$

and

$$\frac{\partial \eta}{\partial t}(x_1,t)=v_2(x_1,R+\eta(x_1,t),t) \quad (1.13)$$

for any $0<x_1<L$, $0<t<T$. Our original aim was to solve just this problem with an unknown interface η, but our attempt has failed.

During the last two decades, however, a substantial beginning has been made on mathematical analysis of the equations governing the motion of compressible and incompressible fluids with free boundary, see e.g. [1], [4], [7], [8], [12], [14], [18], [19], [20], and references therein. Since this free boundary value problem seems to be very difficult to treat, here the Navier–Stokes system on a given time dependent domain coupled with a parabolic equation through the Neumann type boundary condition for the second component of the velocity is studied. We establish an existence result for Problem (1.1)-(1.10) assuming that h is known. For fixed h we also prove an existence result for the limit case letting $\kappa\to\infty$. We reflect the free boundary problem twice. In numerical experiments as well as in proof of local existence result for a special case of the regularized system (Section 3), which includes the problem of finding an unknown interface $h=R+\eta$. The methods, that we apply, borrow material from [21], [10], [2], [11] and the references therein, nevertheless, their application to our problem seems to be not straightforward.

The paper is organized as follows. Our study initially, in Section 2, recalls the original fluid-structure interaction model [19] and explains our main assumptions leading to Problem (1.1)-(1.10). In Section 3 we transform our problem to the problem on a fix domain and we conclude this section by making precise the meaning of a weak solution. To prove the existence of the solution we use the method of semi-discretization in time. The existence of a solution in one time step is determined in Section 4. Sections 5, 6 provide necessary a priori estimates and a convergence theorem to prove the existence of the regularized problem. The uniqueness and continuous dependence on the data for the regularized problem is demonstrated in Section 7. Finally, Section 8 presents the main result of this paper where the existence of a weak solution to Problem (1.1)-(1.10) is proved. The case of fixed κ and also the case of $\kappa\to\infty$ are considered. In Section 9 the way of our approximation is utilized to perform some numerical experiments. At the end, Section 10 provides an explanation of our failed attempt to solve the original problem. Nevertheless, using the result of Section 7 and Banach's fixed point theorem we are able to solve a free boundary value problem for the special case of the regularized system.

2. Motivation and derivation of the model

Should we restrict ourselves to the two-dimensional fluid–structure interaction model proposed by A. Quarteroni in [19],[12], it reads as follows. The problem is to find a function $\eta(x_1,t)$ defined for $0 < x_1 < L$, $0 < t < T$ and functions

$$\boldsymbol{v}(x_1,x_2,t) = (v_1(x_1,x_2,t), v_2(x_1,x_2,t)),\ p(x_1,x_2,t)$$

defined in the domain $\Omega(R+\eta)$, with the following properties:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \operatorname{div}\left(\mu\left(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T\right)\right) - \nabla p\ ,\quad \operatorname{div}\boldsymbol{v}\ = 0 \tag{2.1}$$

in $\Omega(R+\eta)$,

$$\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta - c\frac{\partial^3\eta}{\partial t\partial x_1^2} = H \tag{2.2}$$

for any $0 < x_1 < L$, $0 < t < T$, where

$$H = \frac{1}{\rho E}\left(p - p_w - \mu\left((\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T)\cdot\boldsymbol{n}\right)\cdot\boldsymbol{e}_2\right),\quad \boldsymbol{n} = \left(-\frac{\partial\eta}{\partial x_1}, 1\right) \tag{2.3}$$

and

$$\boldsymbol{v}(x_1, R+\eta(x_1,t), t) = \frac{\partial\eta}{\partial t}\,\frac{\boldsymbol{n}}{|\boldsymbol{n}|}(x_1,t)\ . \tag{2.4}$$

Problem (2.1)–(2.4) is in [19] equipped with the following conditions:

$$\left.\begin{array}{lr}
\eta(0,t) = \alpha(t)\,,\quad \eta(L,t) = \beta(t) & t\in(0,T)\,,\\
\eta(x_1,0) = \eta_0(x_1)\,,\quad \frac{\partial\eta}{\partial t}(x_1,0) = \eta_1(x_1) & x_1\in(0,L)\,,\\
\boldsymbol{v} = \boldsymbol{g} & \text{on}\quad \Gamma_{in}\,,\\
\mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = 0\,,\quad 2\mu\frac{\partial v_1}{\partial x_1} - p + p_{out} = 0 & \text{on}\quad \Gamma_{out}\,,\\
v_2 = 0\,,\quad \mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) = 0 & \text{on}\quad \Gamma_c\,,\\
\boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0(\cdot) &
\end{array}\right\} \tag{2.5}$$

for given functions $\alpha,\ \beta,\ \eta_0,\ \eta_1,\ \boldsymbol{g},\ p_{out}$ and $\boldsymbol{v}_0$.

In what follows we shall simplify and regularize Problem (2.1)–(2.5) in several steps in order to arrive at our approximation given by (1.1)–(1.10).

1. We replace the operator $\operatorname{div}\left(\mu\left(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T\right)\right)$ in (2.1) by $\mu\Delta\boldsymbol{v}$ and correspondingly modify H in (2.2) putting

$$H = \frac{1}{\rho E}\left(p - p_w - \mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial\eta}{\partial x_1}\right) - \mu\frac{\partial v_2}{\partial x_2}\right)\ . \tag{2.6}$$

We take

$$v_1(x_1, R+\eta(x_1,t), t) = 0 \quad\text{ and }\quad v_2(x_1, R+\eta(x_1,t), t) = \frac{\partial\eta}{\partial t}(x_1,t) \tag{2.7}$$

instead of (2.4).

2. Now, following [11], we take $\kappa \gg 1$ and we approximate (2.2) and the second condition of (2.7) by

$$E\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta - c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right] = \kappa\left[v_2 - \frac{\partial\eta}{\partial t}\right] \tag{2.8}$$

and (2.6) by

$$\mu \frac{\partial v_2}{\partial x_1}\left(-\frac{\partial \eta}{\partial x_1}\right) + \mu \frac{\partial v_2}{\partial x_2} - p + p_w - \frac{\rho}{2} v_2 \left(v_2 - \frac{\partial \eta}{\partial t}\right) = \rho\kappa \left[\frac{\partial \eta}{\partial t} - v_2\right] . \tag{2.9}$$

Note that by

$$\kappa \longrightarrow \infty$$

we get, at least formally, that the second relation of (2.7) is satisfied. (1.12) holds then as well. This is the key point of our approximation.

One of the possible physical interpretations for introducing finite κ comes from mathematical modelling of semipervious boundary where a boundary condition of the third type occurs. This boundary condition is encountered, for example, when a clogged (e.g. by a thin layer of silt or clay) river bed serves as a boundary of the flow domain.

In our case, the boundary Γ_{wall} seems to be partly permeable for finite κ, but letting $\kappa \to \infty$ it becomes impervious.

3. Now we split the problem by decoupling the fluid - domain problem assuming that a wall deformation $\eta^{(k)} = \eta^{(k)}(x_1, t)$ and thus, the domain

$$\Omega^{(k)} \equiv \Omega(R + \eta^{(k)})$$

is given. We look then for a solution

$$(\boldsymbol{v}, p, \eta) = (\boldsymbol{v}^{(k+1)}, p^{(k+1)}, \eta^{(k+1)})$$

of the following problem:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \mu \Delta \boldsymbol{v} - \nabla p , \quad \mathrm{div}\ \boldsymbol{v} = 0 \qquad \text{in} \quad \Omega^{(k)} , \tag{2.10}$$

$$\begin{aligned} &\left[\mu \frac{\partial v_2}{\partial x_1}\left(-\frac{\partial \eta^{(k)}}{\partial x_1}\right) + \mu \frac{\partial v_2}{\partial x_2} - p + p_w \right. \\ &\quad \left. - \frac{\rho}{2} v_2 \left(v_2 - \frac{\partial \eta^{(k)}}{\partial t}\right)\right](x_1, R + \eta^{(k)}(x_1, t), t) \\ &= \rho\kappa \left(\lambda \frac{\partial \eta}{\partial t}(x_1, t) + (1-\lambda)\frac{\partial \eta^{(k)}}{\partial t}(x_1, t) - v_2(x_1, R + \eta^{(k)}(x_1, t), t)\right) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &-E\left[\frac{\partial^2 \eta}{\partial t^2} - a\frac{\partial^2 \eta}{\partial x_1^2} + b\eta - c\frac{\partial^3 \eta}{\partial t \partial x_1^2}\right](x_1, t) \\ &= \kappa \left(\lambda \frac{\partial \eta}{\partial t}(x_1, t) + (1-\lambda)\frac{\partial \eta^{(k)}}{\partial t}(x_1, t) - v_2(x_1, R + \eta^{(k)}(x_1, t), t)\right) \end{aligned} \tag{2.12}$$

for any $0 < x_1 < L$, $0 < t < T$ and given $\lambda \in (0, 1]$. Note that in the right hand side of (2.9) we approximate $\dfrac{\partial \eta}{\partial t}$ with $\lambda \dfrac{\partial \eta^{(k+1)}}{\partial t} + (1-\lambda)\dfrac{\partial \eta^{(k)}}{\partial t}$.

4. Boundary and initial conditions for η we simplify by putting $\alpha(t) = \beta(t) \equiv 0$ and $\eta_0(x_1) = \eta_1(x_1) \equiv 0$, see (2.5). The inflow condition (2.5) given by the non homogeneous Dirichlet boundary condition we replace by Neumann type boundary conditions involving pressure as it seems to be more natural to have given pressure impulses $p_{in}(\cdot, t)$ on Γ_{inflow} than the prescribed velocity $\boldsymbol{g}$.

As concerns boundary conditions on Γ_{in} and Γ_{out}, in both cases we prescribe the second component of velocity $v_2 = 0$ and we consider the Neumann type boundary conditions for the first component of the velocity involving the dynamic pressure

$$p + \frac{\rho}{2}|\boldsymbol{v}|^2 \qquad \text{instead of static pressure} \qquad p\ .$$

This type of boundary conditions are considered in [13] and discussed in [19]. Note that in [6] the problem of 3D-flow in a network of pipes is considered and on Γ_{in} they prescribed boundary conditions involving the pressure

$$\boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{and} \qquad p + \frac{\rho}{2}|\boldsymbol{v}|^2 = p_0\ .$$

Boundary conditions on Γ_c correspond to the assumptions of symmetry.

5. Finally, the incompressibility condition $\operatorname{div} \boldsymbol{v} = 0$ in $\Omega(h)$ we approximate in the next section by

$$\varepsilon\left(\frac{\partial p}{\partial t} - \Delta p\right) + \operatorname{div} \boldsymbol{v} = 0 \qquad \text{in} \quad \Omega(h) \tag{2.13}$$

for small $0 < \varepsilon < 1$ and we let $\varepsilon \to 0$ in Section 8. We overcome the difficulties connected with this constraint following [21], where the approximation $\varepsilon\partial_t p + \operatorname{div}\boldsymbol{v} = 0$ was used. Since the divergence free condition does not hold by this approximation, we follow [21] and add the term $\frac{\rho}{2} v_i \operatorname{div} \boldsymbol{v}$ to the nonlinear convective term, see (3.1) below. Later, after passing $\varepsilon \to 0$ we show that $\operatorname{div} \boldsymbol{v} = 0$ holds and this term dissapears.

6. Our idea for constructively solving the original problem (1.1)–(1.10) with (1.4) and (1.5) replaced by (1.12) and (1.13) is to select some $h^{(0)} = R + \eta^{(0)}$ and then iteratively solve the system on a given time dependent domain $\Omega(h^{(k-1)})$ for an unknown $(\boldsymbol{v}^{(k)}, p^{(k)}, \eta^{(k)})$, $k \in \mathbb{N}$. In addition, in the items above we regularize the problem by introducing positive parameters ε, κ and by adding some additional terms that should disappear if we let $\varepsilon \to 0$ and $\kappa \to \infty$. We hope that physically and mathematically correct solutions should arise as the limit of solutions to our regularized system. Unfortunately, we are not able to prove that $\{(\boldsymbol{v}^{(k)}, p^{(k)}, \eta^{(k)})\}_{k=1}^{\infty}$ converges to a solution of the original problem requiring also that $\varepsilon^{(k)} \to 0$ and $\kappa^{(k)} \to \infty$ if $k \to \infty$. Therefore we denote

$$h(x_1, t) = R + \eta^{(k)}(x_1, t) \tag{2.14}$$

and we arrive at our problem (1.1)–(1.10). Numerical experiments indicate, however, that the process converges, see Section 9 below. If $0 < \varepsilon \ll 1$ and $\kappa \gg 1$ are fixed, we show in Section 10 that $\{(\boldsymbol{v}^{(k)}, p^{(k)}, \eta^{(k)})\}_{k=1}^{\infty}$ converges for small data.

3. Auxiliary problem

Given numbers $\kappa \gg 1$, $0 < \varepsilon \ll 1$ and a smooth function h, consider now the system

$$\rho\frac{\partial v_i}{\partial t} + \rho\sum_{i=1}^{2} v_j\frac{\partial v_i}{\partial x_j} + \frac{\rho}{2}\, v_i \,\mathrm{div}\, \boldsymbol{v} = \mu\Delta v_i - \frac{\partial p}{\partial x_i}\ , \qquad i = 1, 2 \tag{3.1}$$

together with (2.13) in $\Omega(h)$,

$$\begin{aligned} &\left[\frac{\partial^2\eta}{\partial t^2} - a\frac{\partial^2\eta}{\partial x_1^2} + b\eta - c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right](x_1, t) \\ &= -\frac{\kappa}{E}\left(\lambda\frac{\partial\eta}{\partial t}(x_1, t) + (1-\lambda)\frac{\partial h}{\partial t}(x_1, t) - v_2(x_1, h(x_1, t), t)\right) \end{aligned} \tag{3.2}$$

for any $0 < x_1 < L$, $0 < t < T$, equipped with the following boundary and initial conditions:

$$v_1(x_1, h(x_1, t), t) = 0$$

$$\left[\mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial h}{\partial x_1}\right) + \mu\frac{\partial v_2}{\partial x_2} - p + p_w - \frac{\rho}{2}v_2\left(v_2 - \frac{\partial h}{\partial t}\right)\right](x_1, h(x_1, t), t)$$

$$= \rho\kappa\left(\lambda\frac{\partial\eta}{\partial t}(x_1, t) + (1-\lambda)\frac{\partial h}{\partial t}(x_1, t) - v_2(x_1, h(x_1, t), t)\right)$$

$$\left[\frac{\partial p}{\partial x_1}\left(-\frac{\partial h}{\partial x_1}\right) + \frac{\partial p}{\partial x_2}\right](x_1, h(x_1, t), t) = -\frac{\rho}{2}\frac{\partial h}{\partial t}(x_1, t)\ p(x_1, h(x_1, t), t)$$

for any $0 < x_1 < L,\ 0 < t < T$,

$$\left(\mu\frac{\partial v_1}{\partial x_1} - p + p_{out} - \frac{\rho}{2}\,|\boldsymbol{v}|^2\right)(L, x_2, t) = 0$$

$$v_2(L, x_2, t) = 0, \qquad \frac{\partial p}{\partial x_1}(L, x_2, t) = 0$$

for any $0 < x_2 < R$, $0 < t < T$,

$$\left(\mu\frac{\partial v_1}{\partial x_1} - p + p_{in} - \frac{\rho}{2}\,|\boldsymbol{v}|^2\right)(0, x_2, t) = 0$$

$$v_2(0, x_2, t) = 0, \qquad \frac{\partial p}{\partial x_1}(0, x_2, t) = 0$$

for any $0 < x_2 < R$, $0 < t < T$,

$$\mu\frac{\partial v_1}{\partial x_2}(x_1, 0, t) = 0\ , \qquad v_2(x_1, 0, t) = 0\ , \qquad \frac{\partial p}{\partial x_2}(x_1, 0, t) = 0$$

for any $0 < x_1 < L$, $0 < t < T$,

$$\boldsymbol{v}(x_1, x_2, 0) = \mathbf{0}\ , \qquad p(x_1, x_2, 0) = 0$$

for any $0 < x_2 < R$, $0 < x_1 < L$,

$$\eta(x_1, 0) = 0\ , \qquad \frac{\partial \eta}{\partial t}(x_1, 0) = 0$$

for any $0 < x_1 < L$ and, finally,

$$\eta(0, t) = \eta(L, t) = 0$$

for any $0 < t < T$. Note that $h(0, t) = h(L, t) = h(x_1, 0) = R$.

By performing the tedious but straightforward formal manipulations one can see that if $(\boldsymbol{v}, p, \eta)$ solves our problem, then

$$\boldsymbol{u}(y_1, y_2, t) \stackrel{\text{def}}{=} \boldsymbol{v}(y_1, h(y_1, t)y_2, t)\ , \tag{3.3}$$

$$q(y_1, y_2, t) \stackrel{\text{def}}{=} \rho^{-1} p(y_1, h(y_1, t)y_2, t) \tag{3.4}$$

and

$$u(y_1, t) \stackrel{\text{def}}{=} \frac{\partial \eta}{\partial t}(y_1, t) \tag{3.5}$$

for $0 < y_1 < L$, $0 < y_2 < 1$, $0 < t < T$ solve the following problem:

$$\begin{aligned} &\frac{\partial (hu_1)}{\partial t} - \frac{\partial h}{\partial t}\frac{\partial (y_2 u_1)}{\partial y_2} + hu_1\left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2}\right) + u_2\frac{\partial u_1}{\partial y_2} \\ &\quad + \frac{h}{2}u_1 \operatorname{div}_h \boldsymbol{u} - \frac{\partial}{\partial y_1}\left[\nu h\left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2}\right) - hq\right] \\ &- \frac{\partial}{\partial y_2}\left[\frac{\nu}{h}\frac{\partial u_1}{\partial y_2} - \nu y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2}\right) + y_2\frac{\partial h}{\partial y_1}q\right] = 0 \end{aligned} \tag{3.6}$$

in $D \times (0, T)$,

$$\begin{aligned} &\frac{\partial (hu_2)}{\partial t} - \frac{\partial h}{\partial t}\frac{\partial (y_2 u_2)}{\partial y_2} + hu_1\left(\frac{\partial u_2}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_2}{\partial y_2}\right) + u_2\frac{\partial u_2}{\partial y_2} \\ &\quad + \frac{h}{2}u_2 \operatorname{div}_h \boldsymbol{u} - \frac{\partial}{\partial y_1}\left[\nu h\left(\frac{\partial u_2}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_2}{\partial y_2}\right)\right] \\ &- \frac{\partial}{\partial y_2}\left[\frac{\nu}{h}\frac{\partial u_2}{\partial y_2} - \nu y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial u_2}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_2}{\partial y_2}\right) - q\right] = 0 \end{aligned} \tag{3.7}$$

in $D \times (0, T)$,

$$\begin{aligned} &\varepsilon\left(\frac{\partial (hq)}{\partial t} - \frac{\partial h}{\partial t}\frac{\partial (y_2 q)}{\partial y_2}\right) - \varepsilon\frac{\partial}{\partial y_1}\left[h\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right] \\ &- \varepsilon\frac{\partial}{\partial y_2}\left[\frac{1}{h}\frac{\partial q}{\partial y_2} - y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right] + h \operatorname{div}_h \boldsymbol{u} = 0 \end{aligned} \tag{3.8}$$

in $D \times (0,T)$, where $D = (0,L) \times (0,1)$ and $\nu = \rho^{-1}\mu$,

$$\mathrm{div}_h\, \boldsymbol{u} \stackrel{\text{def}}{=} \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} + \frac{1}{h}\frac{\partial u_2}{\partial y_2}\ , \tag{3.9}$$

$$\begin{aligned}&\frac{\partial u}{\partial t}(y_1,t) - c\,\frac{\partial^2 u}{\partial y_1^2}(y_1,t) - a\frac{\partial^2}{\partial y_1^2}\int_0^t u(y_1,s)ds + b\int_0^t u(y_1,s)ds\\ &= -\frac{\kappa}{E}\left(\lambda u(y_1,t) + (1-\lambda)\frac{\partial h}{\partial t}(y_1,t) - u_2(y_1,1,t)\right)\end{aligned} \tag{3.10}$$

for any $0 < y_1 < L,\ 0 < t < T$ with the boundary and initial conditions listed below:

$$\begin{aligned}&u_1(y_1,1,t) = 0\ , \qquad (3.11)\\ &\left[\frac{\nu}{h}\frac{\partial u_2}{\partial y_2} - \nu\frac{\partial h}{\partial y_1}\left(\frac{\partial u_2}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_2}{\partial y_2}\right) - q\right](y_1,1,t)\\ &= \left(-q_w + \frac{1}{2}u_2\left(u_2 - \frac{\partial h}{\partial t}\right) - \kappa\left(u_2 - \lambda u - (1-\lambda)\frac{\partial h}{\partial t}\right)\right)(y_1,1,t)\,,\\ &\left[\frac{1}{h}\frac{\partial q}{\partial y_2} - \frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{1}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right](y_1,1,t) \ = \ -\frac{1}{2}\frac{\partial h}{\partial t}(y_1,t)\ q(y_1,1,t)\end{aligned}$$

for any $0 < y_1 < L,\ 0 < t < T,\ q_w(y_1,t) = \rho^{-1}p_w(y_1,t)$,

$$\begin{aligned}&\nu\left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} - q\right)(L,y_2,t) = \left(-q_{out} + \frac{1}{2}\left|\boldsymbol{u}\right|^2\right)(L,y_2,t)\ ,\\ &u_2(L,y_2,t) = 0\ , \qquad \left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)(L,y_2,t) = 0\end{aligned} \tag{3.12}$$

for any $0 < y_2 < 1,\ 0 < t < T,\ q_{out}(y_2,t) = \rho^{-1}p_{out}(Ry_2,t)$,

$$\begin{aligned}&\nu\left(\frac{\partial u_1}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial u_1}{\partial y_2} - q\right)(0,y_2,t) = \left(-q_{in} + \frac{1}{2}\left|\boldsymbol{u}\right|^2\right)(0,y_2,t)\\ &u_2(0,y_2,t) = 0\ , \qquad \left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)(0,y_2,t) = 0\end{aligned} \tag{3.13}$$

for any $0 < y_2 < 1,\ 0 < t < T,\ q_{in}(y_2,t) = \rho^{-1}p_{in}(Ry_2,t)$,

$$\nu\frac{\partial u_1}{\partial y_2}(y_1,0,t) = 0\ , \qquad u_2(y_1,0,t) = 0\ , \qquad \varepsilon\frac{\partial q}{\partial y_2}(y_1,0,t) = 0 \tag{3.14}$$

for any $0 < y_1 < L, 0 < t < T$,

$$\boldsymbol{u}(y_1,y_2,0) = \boldsymbol{0}\ , \qquad q(y_1,y_2,0) = 0 \tag{3.15}$$

for any $0 < y_1 < L, 0 < y_2 < 1$ and, finally,

$$u(y_1,0) = u(0,t) = u(L,t) = 0 \tag{3.16}$$

for any $0 < y_1 < L$ and for any $0 < t < T$.

We continue this section by making precise the meaning of the solution of the problem (3.6)–(3.16). To this end we first define

$$V \equiv \boldsymbol{V} \times H^1(D) \times H_0^1(0,L) \tag{3.17}$$

where

$$\boldsymbol{V} \equiv \left\{ \boldsymbol{w} \in H^1(D)^2 :\ w_1 = 0 \quad \text{on } S_w,\ w_2 = 0 \quad \text{on } S_{in} \cup S_{out} \cup S_c \right\} \tag{3.18}$$

and

$$\begin{aligned} S_w &= \{(y_1,1) :\ 0 < y_1 < L\}\ , \qquad S_{in} = \{(0,y_2) :\ 0 < y_2 < 1\}\ , \\ S_{out} &= \{(L,y_2) :\ 0 < y_2 < 1\}\ , \qquad S_c = \{(y_1,0) :\ 0 < y_1 < L\}\ . \end{aligned}$$

Throughout this and the next sections we assume

$$\begin{aligned} h \in C^1([0,L] \times [0,T]) \cap W^{2,2}((0,L) \times (0,T))\ , \\ q_{in}\ ,\ q_{out} \in L^2(0,T;L^2(0,1))\ ,\ q_w \in L^2(0,T;L^2(0,L))\ . \end{aligned} \tag{3.19}$$

The function spaces we use are rather familiar and we omit the definition, see [22].

Definition 3.1 *We call $(\boldsymbol{u},q,u) \in L^2(0,T;V)$ a weak solution of the initial boundary value problem (3.6)–(3.16) if the following two properties are fulfilled:*
1) $\boldsymbol{u} \in L^\infty(0,T;L^2(D)^2)$, $\dfrac{\partial(h\boldsymbol{u})}{\partial t} \in \left(L^2(0,T;\boldsymbol{V}) \cap L^4(0,T;L^4(D)^2)\right)^*$
$= L^2(0,T;V^*) + L^{4/3}(0,T;L^{4/3}(D)^2)$, *that is,*

$$\int_0^T \left\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\zeta} \right\rangle dt + \int_0^T \int_D h\boldsymbol{u} \cdot \frac{\partial \boldsymbol{\zeta}}{\partial t}\, dt = 0 \tag{3.20}$$

for every test function $\boldsymbol{\zeta} \in L^2(0,T;\boldsymbol{V}) \cap L^4(0,T;L^4(D)^2) \cap H^{1,1}(0,T;L^2(D)^2)$ *with* $\boldsymbol{\zeta}(T) = 0$, $q \in L^\infty(0,T;L^2(D))$, $\dfrac{\partial(hq)}{\partial t} \in L^2(0,T;H^{-1}(D))$ *and* $u \in L^\infty(0,T;H_0^1(0,L)) \cap H^1(0,T;L^2(0,L))$.
2) $(\boldsymbol{u},q,u)$ *satisfies the system of differential equations, that is,*

$$\begin{aligned} &-\int_0^T \left\langle \frac{\partial(h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi} \right\rangle dt \qquad (3.21) \\ &= \int_0^T \left\{ \int_D \left(-\frac{\partial h}{\partial t}\frac{\partial(y_2\boldsymbol{u})}{\partial y_2} \cdot \boldsymbol{\psi} + \left(hu_1 \left(\frac{\partial \boldsymbol{u}}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial \boldsymbol{u}}{\partial y_2} \right) + u_2 \frac{\partial \boldsymbol{u}}{\partial y_2} \right) \cdot \boldsymbol{\psi} \right.\right. \\ &\qquad \left. + \frac{h}{2}\, \boldsymbol{u} \cdot \boldsymbol{\psi}\, \mathrm{div}_h \boldsymbol{u} - h\, q\, \mathrm{div}_h \boldsymbol{\psi} \right) dy \ + \nu(\!(\boldsymbol{u},\boldsymbol{\psi})\!)_h \\ &+ R \int_0^1 \left(\left(q_{out} - \frac{1}{2}|u_1|^2 \right) \psi_1(L,y_2,t) - \left(q_{in} - \frac{1}{2}|u_1|^2 \right) \psi_1(0,y_2,t) \right) dy_2 \\ &+ \int_0^L \left(q_w - \frac{1}{2}u_2 \left(u_2 - \frac{\partial h}{\partial t} \right) + \kappa \left(u_2 - \lambda u - (1-\lambda)\frac{\partial h}{\partial t} \right) \right) \psi_2(y_1,1,t)\, dy_1 \end{aligned}$$

$$+\varepsilon\left\langle \frac{\partial(hq)}{\partial t},\phi\right\rangle - \int_D \left(\varepsilon \frac{\partial h}{\partial t}\frac{\partial(y_2 q)}{\partial y_2}\phi - \, h\,\mathrm{div}_h \boldsymbol{u}\,\phi\right) dy$$
$$+\varepsilon\,(\!(q,\phi)\!)_h + \frac{\varepsilon}{2}\int_0^L \frac{\partial h}{\partial t}(y_1,t)\,q\phi\,(y_1,1,t)\,dy_1$$
$$+\int_0^L \left(\frac{\partial u}{\partial t}\xi + c\,\frac{\partial u}{\partial y_1}\frac{\partial \xi}{\partial y_1} + a\frac{\partial}{\partial y_1}\int_0^t u(y_1,s)ds\frac{\partial \xi}{\partial y_1}\right.$$
$$\left.\left. + b\int_0^t u(y_1,s)ds\;\xi + \frac{\kappa}{E}\left(\lambda u + (1-\lambda)\frac{\partial h}{\partial t} - u_2\right)\xi\right)(y_1,t)\,dy_1\right\} dt$$

for every $(\boldsymbol{\psi},\phi,\xi)\in L^2(0,T;V), \boldsymbol{\psi}\in L^4(0,T;L^4(D)^2)$, *where*

$$(\!(\boldsymbol{u},\boldsymbol{\psi})\!)_h \stackrel{\text{def}}{=} (\!(u_1,\psi_1)\!)_h + (\!(u_2,\psi_2)\!)_h \qquad \text{and} \tag{3.22}$$
$$(\!(q,\phi)\!)_h \stackrel{\text{def}}{=} \int_D \left(\left[h\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_1}\right.$$
$$\left. + \left[\frac{1}{h}\frac{\partial q}{\partial y_2} - y_2\frac{\partial h}{\partial y_1}\left(\frac{\partial q}{\partial y_1} - \frac{y_2}{h}\frac{\partial h}{\partial y_1}\frac{\partial q}{\partial y_2}\right)\right]\frac{\partial \phi}{\partial y_2}\right) dy\,.$$

Note, that (3.20) implies

$$\int_0^\tau \left\langle \frac{\partial(h\boldsymbol{u})}{\partial t},\boldsymbol{\zeta}\right\rangle dt + \int_0^\tau \int_D h\boldsymbol{u}\cdot\frac{\partial \boldsymbol{\zeta}}{\partial t}\,dy\,dt = \int_D h\boldsymbol{u}\cdot\boldsymbol{\zeta}\,(\tau)\,dy \tag{3.23}$$

and that (3.21) holds if T is replaced by τ for almost all $\tau\in(0,T)$.

We will use the following form of the interpolation theorems, which play an important role by proving the existence of the weak solution.

Proposition 3.1 *Let* φ *be any function in* $H^1(D)$ *such that* $\varphi = 0$ *on* S_w *or* $\varphi = 0$ *on* S_c. *Then for any* $p\geq 2$ *and for any number* θ *in the interval*

$$\frac{p-2}{p}\leq\theta\leq 1$$

there exists a constant $C = C(p,\theta)$ *such that*

$$\|\varphi\|_{L^p(D)} \leq C\,\|\nabla\varphi\|^{\theta}_{L^2(D)}\,\|\varphi\|^{1-\theta}_{L^2(D)}\,. \tag{3.24}$$

Moreover, if φ *be any function in* $L^2(0,T;H^1(D))\cap L^\infty(0,T;L^2(D))$ *such that* $\varphi = 0$ *on* S_w *or* $\varphi = 0$ *on* S_c *for almost all* t, *then for any* $p\geq 2$

$$\|\varphi\|_{L^{\frac{2p}{p-2}}(0,T;L^p(D))} \leq C\left[\|\varphi\|_{L^\infty(0,T;L^2(D))}\right]^{\frac{2}{p}}\left[\|\varphi\|_{L^2(0,T;H^1(D))}\right]^{\frac{p-2}{p}}\,. \tag{3.25}$$

Proof. The form of Nierenberg–Gagliardo inequality (3.24) can be found e.g. in [15, Theorem 2.2]. Then (3.25) follows from (3.24) for $\theta = (p-2)/p$ by integration over $(0,T)$. ∎

Proposition 3.2 *Denote $S \equiv S_{in} \cup S_c \cup S_{out} \cup S_w$ and let φ be any function in $H^1(D)$ such that $\varphi = 0$ on S_w or $\varphi = 0$ on S_c. Then for any $r > 1$ there exists a constant $C = C(r)$ such that*

$$\|\varphi\|_{L^r(S)} \le C\,\|\nabla\varphi\|_{L^2(D)}^{1-\frac{1}{r}}\,\|\varphi\|_{L^2(D)}^{\frac{1}{r}}\,. \tag{3.26}$$

Moreover, if φ be any function in $L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))$ such that $\varphi = 0$ on S_w or $\varphi = 0$ on S_c for almost all t then for any $q > 1$

$$\|\varphi\|_{L^{\frac{2q}{q-1}}(0,T;L^q(S))} \le C\left[\|\varphi\|_{L^\infty(0,T;L^2(D))}\right]^{\frac{1}{q}}\left[\|\varphi\|_{L^2(0,T;H^1(D))}\right]^{\frac{q-1}{q}}. \tag{3.27}$$

Proof. The inequality (3.26) can be found e.g. in [15, Inequality (2.21), page 69]. (3.27) follows, similarly as above, from (3.26) by integration over $(0,T)$. ∎

4. Implicit time discretization

To prove the existence of the solution to (3.6)–(3.16) we shall approximate the problem by the perturbed stationary Navier–Stokes systems coupled with the elliptic problems for the pressure and deformation of the wall. This approach is important also for the numerical treatment of the problem. For this we replace

$$\frac{\partial(hu_k)}{\partial t}\,, \quad \frac{\partial(hq)}{\partial t} \quad \text{and} \quad \frac{\partial u}{\partial t}$$

by the backward difference quotients

$$\frac{h^i u_k^i - h^{i-1}u_k^{i-1}}{\Delta t}\,, \qquad \frac{h^i q^i - h^{i-1}q^{i-1}}{\Delta t} \qquad \text{and} \qquad \frac{u^i - u^{i-1}}{\Delta t}$$

for $\Delta t \equiv T/n$, $n \in \mathbb{N}$, $n \gg 1$ and $\int_0^t u(s)ds$ by $\sum_{k=1}^i u^k \Delta t$ for $i\Delta t \le t < (i+1)\Delta t$. Hence, for each $i \in \{1,\dots,n\}$ and given $(\boldsymbol{u}^j, q^j, u^j)$, $0 \le j \le i-1$ we get the perturbed stationary Navier–Stokes system for $\boldsymbol{u}^i$ coupled with the elliptic problems for q^i and u^i, $i = 1,\dots,n$, which in the variational formulation reads as follows:

$$\begin{aligned}
&\int_D \Bigg\{ \left(\frac{h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}}{\Delta t} - d_{\Delta t}h^i\,\frac{\partial(y_2\boldsymbol{u}^i)}{\partial y_2} \right)\cdot\boldsymbol{\omega} \\
&\qquad + \left(h^i u_1^i \left(\frac{\partial \boldsymbol{u}^i}{\partial y_1} - \frac{y_2}{h^i}\frac{\partial h^i}{\partial y_1}\frac{\partial \boldsymbol{u}^i}{\partial y_2} \right) + u_2^i\frac{\partial \boldsymbol{u}^i}{\partial y_2} \right)\cdot\boldsymbol{\omega} + \frac{h^i}{2}\,\boldsymbol{u}^i\cdot\boldsymbol{\omega}\,\mathrm{div}_i\,\boldsymbol{u}^i \\
&\qquad - h^i q^i\,\mathrm{div}_i\,\boldsymbol{\omega} \Bigg\} dy \;+\; \nu\,((\boldsymbol{u}^i,\boldsymbol{\omega}))_{h^i} \;+\; \varepsilon\,((q^i,v))_{h^i} \\
&+ \int_D \left\{ \varepsilon\left(\frac{h^i q^i - h^{i-1}q^{i-1}}{\Delta t} - d_{\Delta t}h^i\,\frac{\partial(y_2 q^i)}{\partial y_2} \right) v + h^i\,v\,\mathrm{div}_i\,\boldsymbol{u}^i \right\} dy \\
&+R\int_0^1 \left\{ \left(q_{out}^i - \frac{1}{2}\left|u_1^i\right|^2 \right)\omega_1\,(L,y_2) - \left(q_{in}^i - \frac{1}{2}\left|u_1^i\right|^2 \right)\omega_1\,(0,y_2) \right\} dy_2
\end{aligned} \tag{4.1}$$

$$
\begin{aligned}
&+\int_0^L \left\{ q_w^i - \frac{1}{2} u_2^i \left(u_2^i - d_{\Delta t} h^i\right) \omega_2 \right. \\
&\qquad \left. + \kappa \left(u_2^i - \lambda u^i - (1-\lambda) d_{\Delta t} h^i\right) \left(\omega_2 - \frac{\vartheta}{E}\right) + \frac{\varepsilon}{2} d_{\Delta t} h^i\, q^i v \right\} (y_1, 1)\, dy_1 \\
&+\int_0^L \left(\frac{u^i - u^{i-1}}{\Delta t} \vartheta + c \frac{\partial u^i}{\partial y_1} \frac{\partial \vartheta}{\partial y_1} + a \sum_{k=1}^{i} \frac{\partial u^k}{\partial y_1} \Delta t \frac{\partial \vartheta}{\partial y_1} + b \left(\sum_{k=1}^{i} u^k \Delta t \right) \vartheta \right) dy_1 \\
&= 0
\end{aligned}
$$

for any $\boldsymbol{\varpi} = (\boldsymbol{\omega}, v, \vartheta) \in V$, where the notation from (3.22) with h^i instead of h is adopted. Here $h^i(y_1) = h(y_1, i\Delta t)$,

$$
d_{\Delta t} h^i = \frac{h^i - h^{i-1}}{\Delta t} \qquad \text{and} \qquad q^i_{\dots}(y_1) = \frac{1}{\Delta t} \int_{(i-1)\Delta t}^{i\Delta t} q_{\dots}(y_1, s)\, ds\ .
$$

In what follows we shall consider the following variational problem:

$$
\text{Find } \boldsymbol{w}^i = (\boldsymbol{u}^i, q^i, u^i) \in V \text{ such that}
$$

$$
a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) + b^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{\varpi}) = L^i(\boldsymbol{\varpi}) \qquad \forall \boldsymbol{\varpi} \in V\ , \tag{4.2}
$$

where $\boldsymbol{\varpi} = (\boldsymbol{\omega}, v, \vartheta)$, V is defined by (3.17) and $a^i(\cdot,\cdot)$, $b^i(\cdot,\cdot,\cdot)$, $L^i(\cdot)$ are determined by (4.1), i.e.

1. $a^i(\cdot,\cdot):\ V \times V \to \mathbb{R}$ be the following bilinear continuous form on V:

$$
\begin{aligned}
a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) \;=\;& \nu\, (\!(\boldsymbol{u}^i, \boldsymbol{\omega})\!)_{h^i} + \varepsilon\, (\!(q^i, v)\!)_{h^i} \\
&+ \frac{1}{\Delta t} \int_D h^i\, \boldsymbol{u}^i \cdot \boldsymbol{\omega}\, dy + \frac{\varepsilon}{\Delta t} \int_D h^i\, q^i\, v\, dy \\
&+ \int_0^L \left((c + a\Delta t) \frac{\partial u^i}{\partial y_1} \frac{\partial \vartheta}{\partial y_1} + \left(\frac{1}{\Delta t} + b\Delta t \right) u^i \vartheta \right) dy_1 \\
&- \int_D d_{\Delta t} h^i\, \frac{\partial (y_2 \boldsymbol{u}^i)}{\partial y_2} \cdot \boldsymbol{\omega}\, dy + \int_0^L \frac{1}{2} u_2^i d_{\Delta t} h^i \omega_2\ (y_1, 1)\, dy_1 \\
&- \varepsilon \int_D d_{\Delta t} h^i\, \frac{\partial (y_2 q^i)}{\partial y_2} v\, dy + \frac{\varepsilon}{2} \int_0^L d_{\Delta t} h^i\, q^i\, v\, (y_1, 1)\, dy_1 \\
&+ \kappa \int_0^L (\lambda u^i - u_2^i) \left(\frac{\vartheta}{E} - \omega_2 \right) (y_1)\, dy_1 \\
&+ \int_D \left(h^i\ v\, \mathrm{div}_i\, \boldsymbol{u}^i - h^i q^i\, \mathrm{div}_i\, \boldsymbol{\omega} \right) dy\ .
\end{aligned}
$$

2. The trilinear form $b^i(\cdot,\cdot,\cdot)$ is defined by

$$
b^i(\cdot,\cdot,\cdot):\ V \times V \times V \longrightarrow \mathbb{R}\,,
$$

$$
b^i(\boldsymbol{w}^i, \boldsymbol{m}^i, \boldsymbol{\varpi}) \;=\; B^i(\boldsymbol{u}^i, \boldsymbol{z}^i, \boldsymbol{\omega}) + \int_D \frac{h^i}{2}\, \boldsymbol{z}^i \cdot \boldsymbol{\omega}\, \mathrm{div}_i\, \boldsymbol{u}^i\, dy \tag{4.3}
$$

$$
\begin{aligned}
& -\frac{R}{2}\int_0^1 u_1^i z_1^i \omega_1(L,y_2)\,dy_2 + \frac{R}{2}\int_0^1 u_1^i z_1^i \omega_1(0,y_2)\,dy_2 \\
& -\frac{1}{2}\int_0^L u_2^i z_2^i \omega_2(y_1,1)\,dy_1\,, \\
B^i(\boldsymbol{u}^i,\boldsymbol{z}^i,\boldsymbol{\omega}) \quad = \quad & \int_D \left(h^i u_1^i \left(\frac{\partial \boldsymbol{z}^i}{\partial y_1} - \frac{y_2}{h^i}\frac{\partial h^i}{\partial y_1}\frac{\partial \boldsymbol{z}^i}{\partial y_2}\right) + u_2^i \frac{\partial \boldsymbol{z}^i}{\partial y_2}\right)\cdot \boldsymbol{\omega}\,dy
\end{aligned}
$$

for $\boldsymbol{m}^i = (\boldsymbol{z}^i,\cdot,\cdot)$, $\boldsymbol{w}^i = (\boldsymbol{u}^i,\cdot,\cdot)$, $\boldsymbol{\varpi} = (\boldsymbol{\omega},\cdot,\cdot)$ and $\boldsymbol{z}^i,\boldsymbol{u}^i,\boldsymbol{\omega}\in\boldsymbol{V}$.

3. Finally, $L^i(\cdot)$ is the linear functional on V,

$$
\begin{aligned}
L^i(\boldsymbol{\varpi}) \quad = \quad & \frac{1}{\Delta t}\int_D h^{i-1}\left(\boldsymbol{u}^{i-1}\cdot\boldsymbol{\omega} + \varepsilon\, q^{i-1}\, v\right)\,dy + \frac{1}{\Delta t}\int_0^L u^{i-1}\vartheta\,dy_1 \\
& + R\int_0^1 \left(q_{in}^i\,\omega_1\,(0,y_2) - q_{out}^i\,\omega_1\,(L,y_2)\right)\,dy_2 \\
& + \int_0^L \left(-q_w^i\,\omega_2\,(y_1,1) - \sum_{k=1}^{i-1}\left(a\frac{\partial u^k}{\partial y_1}\frac{\partial \vartheta}{\partial y_1} + b u^k \vartheta\right)(y_1)\Delta t\right)\,dy_1 \\
& + \kappa(1-\lambda)\int_0^L d_{\Delta t}h^i(\omega_2 - \frac{\vartheta}{E})(y_1)dy_1\,.
\end{aligned}
$$

Theorem 4.1 *Let $i\in\{1,2,\dots,n\}$ and $\boldsymbol{w}^j\in V$ for $j\le i-1$ be given. Assume there are nonnegative constants α, K, independent on i, such that*

$$
0<\alpha\le h^i(y_1)\le \alpha^{-1} \tag{4.4}
$$

and

$$
\left|\frac{\partial h^i}{\partial y_1}(y_1)\right| + \left|d_{\Delta t}h^i(y_1)\right| \le K \tag{4.5}
$$

for all $0\le y_1\le L$ and $i=1,2,\dots n$. Moreover, assume that

$$
q_{in}^i, q_{out}^i \in L^2(0,1),\ q_w^i\in L^2(0,L) \quad \text{and} \quad \Delta t\le \alpha/K\ .
$$

Then Problem (4.2) has at least one solution.

Proof. Following [6, Proof of Theorem 2.1], we use Galerkin's method. V is a closed subspace of $H^1(D)^3\times H_0^1(0,L)$ and it is thus possible to choose a basis $\{\boldsymbol{\zeta}_k\}_{k=1}^\infty\subset V$. For every $\ell\ge 1$ we define an approximate problem by:

$$
\text{Find } c_{k\ell}\in\mathbb{R},\ 1\le k\le \ell \text{ such that } \boldsymbol{w}_\ell = \sum_{k=1}^{\ell} c_{k\ell}\,\boldsymbol{\zeta}_k \text{ is a solution of}
$$

$$
a^i(\boldsymbol{w}_\ell,\boldsymbol{\zeta}_k) + b^i(\boldsymbol{w}_\ell,\boldsymbol{w}_\ell,\boldsymbol{\zeta}_k) = L^i(\boldsymbol{\zeta}_k) \qquad \forall k=1,\dots,\ell\ . \tag{4.6}
$$

To prove the existence of a solution to (4.6) we will use the following lemma (see [16, Lemma 1.4.3, p.53] or [21, Lemma 2.1.4, p. 164]).

Lemma 4.2 *Let Y be a finite–dimensional Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let P be a continuous mapping from Y into itself such that, for a sufficiently large $\rho > 0$,*

$$(P(\zeta),\zeta) \geq 0 \qquad \forall \zeta \in Y \text{ such that } \|\zeta\| = \rho \,. \tag{4.7}$$

Then there exists $\zeta \in Y$, $\|\zeta\| \leq \rho$ such that $P(\zeta) = 0$.

In our case $Y = Y_\ell = \mathrm{span}\{\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_\ell\}$ equipped with the scalar product of $H^1(D)^3 \times H_0^1(0,L)$, and for any $\boldsymbol{\zeta} \in Y$ define $P(\boldsymbol{\zeta}) = P_\ell(\boldsymbol{\zeta}) \in Y$ using Riesz's Representation Theorem by

$$(P_\ell(\boldsymbol{\zeta}), \boldsymbol{z}) = a^i(\boldsymbol{\zeta}, \boldsymbol{z}) + b^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}, \boldsymbol{z}) - L^i(\boldsymbol{z}) \qquad \forall \boldsymbol{z} \in Y_\ell \,.$$

It is easy to see that P_ℓ is continuous. In order to prove (4.7) let us first introduce the following lemma.

Lemma 4.3 *Let (4.4)–(4.5) be satisfied. Then*

$$(\!(v,v)\!)_{h^i} \geq \frac{\alpha}{2+K^2} \int_D |\nabla v|^2 \, dy \tag{4.8}$$

for any $v \in H^1(D)$, where $(\!(\cdot,\cdot)\!)_{h^i}$ is given by (3.22) with h^i instead of h.

Proof. Note first that

$$(\!(v,v)\!)_{h^i} = \int_D \left\{ h^i \left(\frac{\partial v}{\partial y_1} - \frac{y_2}{h^i} \frac{\partial h^i}{\partial y_1} \frac{\partial v}{\partial y_2} \right)^2 + \frac{1}{h^i} \left(\frac{\partial v}{\partial y_2} \right)^2 \right\} dy \,. \tag{4.9}$$

For a moment, let us denote

$$A = \sqrt{h^i} \frac{\partial v}{\partial y_1} \,, \; B = \frac{1}{\sqrt{h^i}} \frac{\partial v}{\partial y_2} \,, \; z = y_2 \frac{\partial h}{\partial y_1}$$

and rewrite

$$(\!(v,v)\!)_{h^i} = \int_D \Big\{ A^2 - 2zAB + (1+z^2)B^2 \Big\} dy \,. \tag{4.10}$$

Taking $0 < \delta < 1/(K^2+1)$ one easily obtains

$$A^2 - 2zAB + (1+z^2)B^2 \geq \delta A^2 + \frac{1-\delta(z^2+1)}{1-\delta} B^2 .$$

As $|z| < K$, putting $\delta = 1/(K^2+2)$, (4.8) follows easily. ■

Next, note that

$$(P_\ell(\boldsymbol{\zeta}), \boldsymbol{\zeta}) = a^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}) - L^i(\boldsymbol{\zeta}) \,,$$

as

$$b^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}, \boldsymbol{\zeta}) \; = \; 0 \tag{4.11}$$

for $\boldsymbol{\zeta} = (\boldsymbol{v}, p, \lambda E v)$. To get (4.11) we have used the fact that

$$b^i(\boldsymbol{w}^i, \boldsymbol{m}^i, \boldsymbol{\varpi}) = \frac{1}{2} B^i(\boldsymbol{u}^i, \boldsymbol{z}^i, \boldsymbol{\omega}) - \frac{1}{2} B^i(\boldsymbol{u}^i, \boldsymbol{\omega}, \boldsymbol{z}^i) \tag{4.12}$$

for $\boldsymbol{m}^i = (\boldsymbol{z}^i, \cdot, \cdot)$, $\boldsymbol{w}^i = (\boldsymbol{u}^i, \cdot, \cdot)$, $\boldsymbol{\varpi} = (\boldsymbol{\omega}, \cdot, \cdot)$, where B^i is defined by (4.3). Some calculations have to be performed to obtain (4.12) and we omit it here. Next, it is easy to see that there exists a positive constant C such that

$$\left| L^i(\boldsymbol{\zeta}) \right| \leq C \|\boldsymbol{\zeta}\|_V \qquad \forall \boldsymbol{\zeta} \in V \; . \tag{4.13}$$

Finally, one can easily verify that

$$\begin{aligned} a^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \;&=\; \nu(\!(\boldsymbol{v}, \boldsymbol{v})\!)_{h^i} + \varepsilon(\!(p, p)\!)_{h^i} \\ &+\; \int_D \left[\frac{h^i}{\Delta t} + \frac{1}{2} d_{\Delta t} h^i \right] \left(|\boldsymbol{v}|^2 + \varepsilon |p|^2 \right) \, dy \\ &+\; \int_0^L \lambda E (c + a\Delta t) \left| \frac{\partial v}{\partial y_1} \right|^2 + \lambda E \left(\frac{1}{\Delta t} + b\Delta t \right) |v|^2 + \kappa (v_2 - \lambda v)^2 \, dy_1 \end{aligned} \tag{4.14}$$

and therefore, if Δt is sufficiently small, say $0 < \Delta t < \alpha/K$, there exists a constant $\delta > 0$ such that

$$a^i(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq \delta \|\boldsymbol{\zeta}\|_V^2 \qquad \forall \boldsymbol{\zeta} \in V \; . \tag{4.15}$$

Hence

$$(P_\ell(\boldsymbol{\zeta}), \boldsymbol{\zeta}) \geq \delta \|\boldsymbol{\zeta}\|_V \left(\|\boldsymbol{\zeta}\|_V - \frac{C}{\delta} \right) \qquad \forall \boldsymbol{\zeta} \in Y_\ell$$

which implies that P_ℓ satisfies (4.7) with $\rho = C/\delta$. Thus, for any $\ell \in \mathbb{N}$ there exists a solution $\boldsymbol{w}_\ell$ of (4.6) which verifies

$$\|\boldsymbol{w}_\ell\|_V \leq \rho \; .$$

Therefore, there exists $\boldsymbol{w}^i \in V$ and a subsequence $\{\boldsymbol{w}_{\ell'}\}$ of $\{\boldsymbol{w}_\ell\}$ such that

$$\boldsymbol{w}_{\ell'} \;\to \boldsymbol{w}^i \text{ weakly in } V \text{ as } \ell' \to \infty \, . \tag{4.16}$$

Due to the compact embedding of V into corresponding Lebesgue spaces $(L^p(\cdot))^3 \times L^2(0, L)$ we get

$$\boldsymbol{w}_{\ell'} \;\to \boldsymbol{w}^i \;\text{ strongly in }\; (L^p(D))^3 \times L^p(0, L) \;\text{ as }\; \ell' \to \infty \tag{4.17}$$

and

$$\boldsymbol{w}_{\ell'} \;\to \boldsymbol{w}^i \;\text{ strongly in }\; (L^p(S))^3 \times L^p(0, L) \text{ as } \; \ell' \to \infty \tag{4.18}$$

for any $p \geq 1$. Concerning test functions, let $\boldsymbol{\varpi} \in V$ and a sequence $\{\boldsymbol{\varpi}_\ell\}$ be such that $\boldsymbol{\varpi}_\ell \in Y_\ell$ and

$$\boldsymbol{\varpi}_\ell \;\to \boldsymbol{\varpi} \;\text{ strongly in }\; V \tag{4.19}$$

as $\ell \to \infty$. Note that it converges also in the spaces as in (4.17) and (4.18). Finally, according to (4.2) for every ℓ' we have

$$a^i(\boldsymbol{w}_{\ell'}, \boldsymbol{\varpi}_{\ell'}) + b^i(\boldsymbol{w}_{\ell'}, \boldsymbol{w}_{\ell'}, \boldsymbol{\varpi}_{\ell'}) = L^i(\boldsymbol{\varpi}_{\ell'}) \,. \tag{4.20}$$

Using (4.16)–(4.19) we can pass to the limit in (4.20) and we obtain that

$$a^i(\boldsymbol{w}^i, \boldsymbol{\varpi}) + b^i(\boldsymbol{w}^i, \boldsymbol{w}^i, \boldsymbol{\varpi}) = L^i(\boldsymbol{\varpi}) \qquad\qquad \forall \boldsymbol{\varpi} \in V \,,$$

i.e. (4.2) holds, which completes the proof. ■

5. A priori estimates

We now ascertain the a priori estimates for $\boldsymbol{u}^i$, q^i, u^i, $i = 1, 2, \ldots, n$. In the first step, we test (4.1) with $\boldsymbol{w}^i = (\boldsymbol{u}^i, q^i, Eu^i)$ and sum over $i = 1, 2, \ldots, r$ for some $r \le n$. Before writing a final formula, let us focus on the separate terms, where some calculations were performed:

$$2\sum_{i=1}^{r} \int_D \left(h^i \boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}\right) \cdot \boldsymbol{u}^i dy \;=\; \int_D h^r \left|\boldsymbol{u}^r\right|^2 dy$$
$$+ \sum_{i=1}^{r} \int_D \left\{ \frac{1}{h^i} \left|h^i \boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}\right|^2 + \frac{h^{i-1}}{h^i} \left(h^i - h^{i-1}\right) \left|\boldsymbol{u}^{i-1}\right|^2 \right\} dy \,,$$

$$-2 \int_D d_{\Delta t} h^i \, \frac{\partial(y_2 \boldsymbol{u}^i)}{\partial y_2} \cdot \boldsymbol{u}^i \, dy$$
$$= \int_0^L d_{\Delta t} h^i \, \left|u_2^i\right|^2 (y_1, 1)\, dy_1 - \int_D d_{\Delta t} h^i \left|\boldsymbol{u}^i\right|^2 dy \,,$$

$$2\sum_{i=1}^{r} \int_0^L \left(u^i - u^{i-1}\right) u^i \, dy_1 = \int_0^L |u^r|^2 \, dy_1 + \sum_{i=1}^{r} \int_0^L \left|u^i - u^{i-1}\right|^2 dy_1 \;,$$

$$U^0 \equiv 0 \,, \qquad U^i \equiv \sum_{k=1}^{i} u^k \Delta t \,, \qquad \frac{U^i - U^{i-1}}{\Delta t} = u^i \,, \tag{5.1}$$

$$a \sum_{i=1}^{r} \int_0^L \frac{\partial U^i}{\partial y_1} \frac{\partial u^i}{\partial y_1} dy_1 \Delta t = \frac{a}{2} \int_0^L \left\{ \left|\frac{\partial U^r}{\partial y_1}\right|^2 dy_1 + \sum_{i=1}^{r} \left|\frac{\partial (U^i - U^{i-1})}{\partial y_1}\right|^2 \right\} dy_1,$$

$$b \sum_{i=1}^{r} \int_0^L U^i u^i \, dy_1 \, \Delta t = \frac{b}{2} \int_0^L \left\{ |U^r|^2 \, dy_1 + \sum_{i=1}^{r} \left|U^i - U^{i-1}\right|^2 \right\} dy_1$$

and, finally, let us recall (4.11). (4.1) and (5.1) then easily yield

$$\int_D h^r \left(|\boldsymbol{u}^r|^2 + \varepsilon \, |q^r|^2 \right) \, dy + E \int_0^L |u^r|^2 \, dy_1$$
$$+ \sum_{i=1}^{r} \int_D \frac{1}{h^i} \left(\left|h^i \boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}\right|^2 \; + \; \varepsilon \left|h^i q^i - h^{i-1} q^{i-1}\right|^2 \right) \, dy$$

$$
\begin{aligned}
&+\sum_{i=1}^{r}\left[E\int_0^L \left|u^i-u^{i-1}\right|^2 dy_1 + 2\nu(\!(\boldsymbol{u}^i,\boldsymbol{u}^i)\!)_{h^i} \;+\; 2\varepsilon(\!(q^i,q^i)\!)_{h^i}\right]\Delta t\\
&+2\sum_{i=1}^{r}\int_0^L \left\{(1-\lambda)\kappa\left(u_2^i-d_{\Delta t}h^i\right)\left(u_2^i-u^i\right)+\lambda\kappa\left|u_2^i-u^i\right|^2\right\} dy_1\,\Delta t\\
&+2cE\sum_{i=1}^{r}\int_0^L\left|\frac{\partial u^i}{\partial y_1}\right|^2 dy_1\Delta t + aE\int_0^L\left\{\left|\frac{\partial U^r}{\partial y_1}\right|^2+\sum_{i=1}^{r}\left|\frac{\partial (U^i-U^{i-1})}{\partial y_1}\right|^2\right\}dy_1\\
&+bE\int_0^L\left\{\left|U^r\right|^2 dy_1+\sum_{i=1}^{r}\int_0^L\left|U^i-U^{i-1}\right|^2\right\}dy_1\\
&=\sum_{i=1}^{r}\left[2R\int_0^1\left(q_{in}^i(y_2)\,u_1^i(0,y_2)-q_{out}^i(y_2)\,u_1^i(L,y_2)\right)dy_2\right.\\
&\left.+2\int_0^L q_w^i(y_1)\,u_2^i(y_1,1)\,dy_1-\int_D\frac{h^{i-1}}{h^i}\left(d_{\Delta t}h^i\right)\left(\left|\boldsymbol{u}^{i-1}\right|^2+\varepsilon\left|q^{i-1}\right|^2\right)dy\right]\Delta t\,.
\end{aligned}
$$

Now, with the assistance of Lemma 4.3 and Young's inequality we obtain

$$
\begin{aligned}
&\int_D h^r\left(|\boldsymbol{u}^r|^2+\varepsilon\,|q^r|^2\right)dy + E\int_0^L |u^r|^2\,dy_1\\
&+\sum_{i=1}^{r}\left[\frac{2\alpha}{2+K^2}\int_D\left(\nu\left|\nabla\boldsymbol{u}^i\right|^2+\varepsilon\left|\nabla q^i\right|^2\right)dy+2cE\int_0^L\left|\frac{\partial u^i}{\partial y_1}\right|^2 dy_1\right]\Delta t\\
&+aE\int_0^L\left|\frac{\partial U^r}{\partial y_1}\right|^2 dy_1 + bE\int_0^L |U^r|^2\,dy_1\\
&\le\;\sum_{i=1}^{r}\left[\max_{0\le y_1\le L}\frac{h^{i-1}}{(h^i)^2}\left[-d_{\Delta t}h^i\right]_+\int_D h^i\left(|\boldsymbol{u}^i|^2+\varepsilon|q^i|^2\right)dy\right.\\
&+C_1\left(\int_D\left|\nabla\boldsymbol{u}^i\right|^2 dy\right)^{1/2}\left(\left\|q_{in}^i\right\|_{L^2(0,1)}+\left\|q_{out}^i\right\|_{L^2(0,1)}+\left\|q_w^i\right\|_{L^2(0,L)}\right)\\
&\left.+\lambda^{-1}(1-\lambda)^2\kappa\int_0^L\left|u_2^i-d_{\Delta t}h^i\right|^2 dy_1\right]\Delta t\,,
\end{aligned}
$$

where C_1 depends only on D. Furthermore,

$$
\begin{aligned}
&\int_D h^r\left(|\boldsymbol{u}^r|^2+\varepsilon\,|q^r|^2\right)dy + E\int_0^L |u^r|^2\,dy_1 \qquad (5.2)\\
&+\sum_{i=1}^{r}\left[\frac{\alpha\nu}{2+K^2}\int_D\left(\left|\nabla\boldsymbol{u}^i\right|^2+\frac{\varepsilon}{\nu}\left|\nabla q^i\right|^2\right)dy+2cE\int_0^L\left|\frac{\partial u^i}{\partial y_1}\right|^2 dy_1\right]\Delta t\\
&\le\;\sum_{i=1}^{r}\left(H_n\int_D h^i\left(|\boldsymbol{u}^i|^2+\varepsilon|q^i|^2\right)dy\;+\frac{C_2^2(2+K^2)}{2\alpha\nu}\left\|q_s^i\right\|^2\right.\\
&\qquad\left.+\frac{2(1-\lambda)^2\kappa}{\lambda}\left\|d_{\Delta t}h^i\right\|^2+\frac{C^2(2+K^2)(1-\lambda)^2\kappa}{\alpha^2\lambda}\int_D h^i\left|\boldsymbol{u}^i\right|^2 dy\right)\Delta t
\end{aligned}
$$

where the constant C comes from (3.26),

$$\left\|q_S^i\right\|^2 \equiv \left\|q_{in}^i\right\|^2_{L^2(0,1)} + \left\|q_{out}^i\right\|^2_{L^2(0,1)} + \left\|q_w^i\right\|^2_{L^2(0,L)}, \quad \left\|d_{\Delta t}h^i\right\| \equiv \left\|d_{\Delta t}h^i\right\|_{L^2(0,L)}$$

and

$$H_n \equiv \max_{1\le i\le n} \max_{0\le y_1\le L} \left[-\frac{h^{i-1}}{(h^i)^2} \left(d_{\Delta t}h^i\right)(y_1) \right]_+ .$$

Next we rewrite (5.2) into the form

$$\zeta_n(t) \le R_n \int_0^t \zeta_n(s)ds + M \int_0^t f_n(s)ds, \tag{5.3}$$

where

$$\zeta_n(t) = \int_D h^i \left(\left|\boldsymbol{u}^i\right|^2 + \varepsilon \left|q^i\right|^2 \right) dy + E \int_0^L |u^i|^2 dy_1 \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t]$$

$i = 0, 1, 2, \dots, n$,

$$R_n \equiv H_n + \frac{C^2(2+K^2)(1-\lambda)^2\kappa}{\alpha^2\lambda}, \quad M \equiv \frac{(2+K^2)C_2^2}{2\alpha\nu} + \frac{2(1-\lambda)\kappa}{\lambda},$$

and

$$f_n(t) = \left\|q_S^i\right\|^2 + (1-\lambda)\left\|d_{\Delta t}h^i\right\|^2 \quad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t],\ i = 0, 1, 2, \dots, n .$$

Thus, by applying Gronwall's lemma, see (7.7) and (7.8) below, we obtain

$$\zeta_n(t) \le M \int_0^t f_n(s)ds\; e^{R_n t}$$

for a.e. $t \in [0, T]$, and the first part of the following theorem follows easily.

Theorem 5.1 *The following a priori estimates hold:*

$$\begin{aligned}
&\max_{1\le r\le n} \left[\int_D h^r \left(|\boldsymbol{u}^r|^2 + \varepsilon |q^r|^2 \right) dy + E \int_0^L |u^r|^2 dy_1 \right] \qquad (5.4)\\
&+ \max_{1\le r\le n} \left[a \int_0^L \left| \frac{\partial U^r}{\partial y_1} \right|^2 dy_1 + b \int_0^L |U^r|^2 dy_1 \right]\\
&+ \sum_{i=1}^n \int_D \frac{1}{h^i} \left[\left|h^i\boldsymbol{u}^i - h^{i-1}\boldsymbol{u}^{i-1}\right|^2 + \varepsilon \left|h^i q^i - h^{i-1}q^{i-1}\right|^2 \right] dy\\
&+ \sum_{i=1}^n \left(\int_0^L E\left|u^i - u^{i-1}\right|^2 + \lambda\kappa \left|u_2^i - u^i\right|^2 \Delta t \right) dy_1\\
&+ \sum_{i=1}^n \left[\frac{\alpha\nu}{2+K^2} \int_D \left(\left|\nabla\boldsymbol{u}^i\right|^2 + \frac{\varepsilon}{\nu}\left|\nabla q^i\right|^2 \right) dy \;+\; c \int_0^L \left| \frac{\partial u^i}{\partial y_1} \right|^2 \right] dy_1\, \Delta t\\
&\le\; P \sum_{i=1}^n \left(\left\|q_S^i\right\|^2 + (1-\lambda)\left\|d_{\Delta t}h^i\right\|^2_{L^2(0,L)} \right) \Delta t,
\end{aligned}$$

where $P = Me^{R_n T}$, $T \equiv n\Delta t$ and $M,\ R_n,\ H_n$ are given above. Note that

$$H_n \longrightarrow \max_{0\le y_1\le L, 0\le t\le T}\left[-\frac{1}{h(y_1,t)}\frac{\partial h}{\partial t}(y_1,t)\right]_+ \quad \text{as } n\to 0\,. \tag{5.5}$$

Moreover,

$$\int_0^L \left\{ \sum_{i=1}^n \left|\frac{u^i-u^{i-1}}{\Delta t}\right|^2 \Delta t + c\left(\left|\frac{\partial u^r}{\partial y_1}\right|^2 + \frac12 \sum_{i=1}^n \left|\frac{\partial\left(u^i-u^{i-1}\right)}{\partial y_1}\right|^2\right)\right\} dy_1$$
$$\le C\sum_{i=1}^n \left(\left\|q_S^i\right\|^2 + (1-\lambda)\left\|d_{\Delta t}h^i\right\|^2_{L^2(0,L)}\right)\Delta t\,, \tag{5.6}$$

where C depends on $M,\ R_n,\ H_n,\ a,\ b,\ c,\ E,\ T$.

Proof. To prove (5.6), we test (4.1) with $\left(\mathbf{0},0,Ed_{\Delta t}u^i\right)$, where $d_{\Delta t}u^i \equiv (u^i-u^{i-1})/\Delta t$ and sum over $i=1,2,\ldots,r$, i.e.,

$$\int_0^L\left\{E\sum_{i=1}^r \left|d_{\Delta t}u^i\right|^2 dy_1\Delta t + \frac{cE}{2}\left|\frac{\partial u^r}{\partial y_1}\right|^2 + \frac{cE}{2}\sum_{i=1}^r\left|\frac{\partial u^i}{\partial y_1} - \frac{\partial u^{i-1}}{\partial y_1}\right|^2\right\}dy_1$$
$$= \sum_{i=1}^r \int_0^L \left\{ -bEU^i(u^i-u^{i-1}) - aE\frac{\partial U^i}{\partial y_1}\frac{\partial(u^i-u^{i-1})}{\partial y_1} \right. \tag{5.7}$$
$$\left. +\kappa\left[\lambda\left(u_2^i(y_1,1)-u^i(y_1)\right) + (1-\lambda)\left(u_2^i(y_1,1) - d_{\Delta t}h^i(y_1)\right)\right] d_{\Delta t}u^i\Delta t\right\}dy_1\,.$$

Next, by applying the discrete per partes and Young's inequality to the right hand side of (5.7) it is not difficult to see that it can be estimated by

$$\int_0^L\left(\frac{bE}{2}\left(|U^r|^2+|u^r|^2\right) + \frac{a^2E}{4c}\left|\frac{\partial U^r}{\partial y_1}\right|^2 + \frac{cE}{4}\left|\frac{\partial u^r}{\partial y_1}\right|^2\right)dy_1$$
$$+\int_0^L\left(\sum_{i=1}^r\left\{bE\left|u^i\right|^2 + aE\left|\frac{\partial u^i}{\partial y_1}\right|^2 + \frac{\kappa^2\lambda^2}{E}\left|u_2^i(y_1,1)-u^i(y_1)\right|^2\right.\right.$$
$$\left.\left. + \frac{\kappa^2(1-\lambda)^2}{E}\left(\left|\boldsymbol{u}^i\right|^2 + \left|d_{\Delta t}h^i\right|^2\right) + \frac{E}{2}\left|d_{\Delta t}u^i\right|^2\right\}\Delta t\right)dy_1$$

and the estimate (5.6) follows at once making use of the estimate (5.4) above. ■

In the sequel the following estimate will be essential to get a priori estimate in the time variable t.

Theorem 5.2 *There exists a non-negative constant C such that*

$$\sum_{i=1}^{n-k}\int_D \left(\left|h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i\right|^2 + \varepsilon\left|h^{i+k}q^{i+k} - h^iq^i\right|^2\right)dy\,\Delta t \ \le\ Ck\Delta t \tag{5.8}$$

for any $1\le k<n$. The constant C does not depend on k,n.

Proof. 1. Recalling the definition of the weak solution let us fix $i \in \{1,\dots,n-k\}$ and add (4.1) through $j = i+1,\dots,i+k$ for fix test functions

$$\boldsymbol{\omega} = h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i \ , \quad v = h^{i+k}q^{i+k} - h^i q^i \ , \quad \vartheta = 0 \ .$$

Then we sum the equality over $i = 1,\dots,n-k$, multiply by Δt and we arrive at

$$\begin{aligned}
&\sum_{i=1}^{n-k}\int_D \left(\left|h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i\right|^2 + \varepsilon\left|h^{i+k}q^{i+k} - h^i q^i\right|^2\right) dy\ \Delta t \\
&= (\Delta t)^2 \sum_{i=1}^{n-k}\sum_{j=i+1}^{i+k}\Bigg\{\int_D d_{\Delta t}h^j \frac{\partial}{\partial y_2}\Big(y_2\boldsymbol{u}^j\Big)\cdot(h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i)\, dy \\
&\qquad - B^j(\boldsymbol{u}^j,\boldsymbol{u}^j,h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i) - \nu(\!(\boldsymbol{u}^j, h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i)\!)_{h^j} \\
&\qquad - \int_D \frac{h^j}{2}\boldsymbol{u}^j \operatorname{div}_j(\boldsymbol{u}^j)\cdot(h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i)\, dy \\
&\qquad + \int_D h^j g^j \operatorname{div}_j(h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i)\, dy - \varepsilon(\!(q^j, h^{i+k}q^{i+k} - h^i q^i)\!)_{h^j} \\
&\qquad + \varepsilon\int_D d_{\Delta t}h^j\frac{\partial}{\partial y_2}\Big(y_2 q^j\Big)(h^{i+k}q^{i+k} - h^i q^i)\, dy \\
&\qquad + \int_D h^j \operatorname{div}_j(\boldsymbol{u}^j)(h^{i+k}q^{i+k} - h^i q^i)\, dy \\
&\qquad - R\int_0^1 (q^j_{out} - \frac{1}{2}|u_1^j|^2)(h^{i+k}u_1^{i+k} - h^i u_1^i)\ (L,y_2)\, dy_2 \\
&\qquad + R\int_0^1 (q^j_{in} - \frac{1}{2}|u_1^j|^2)(h^{i+k}u_1^{i+k} - h^i u_1^i)\ (0,y_2)\, dy_2 \\
&\qquad + \int_0^L \Big(q^j_w - \frac{1}{2}u_2^j(u_2^j - d_{\Delta t}h^j)\Big)(h^{i+k}u_2^{i+k} - h^i u_2^i)(y_1,1)\, dy_1 \\
&\qquad + \kappa\int_0^L \Big(\lambda(u_2^j - u^j) + (1-\lambda)(u_2^j - d_{\Delta t}h^j)\Big)(h^{i+k}u_2^{i+k} - h^i u_2^i)\, dy_1 \\
&\qquad - \frac{\varepsilon}{2}\int_0^L d_{\Delta t}h^j q^j(h^{i+k}q^{i+k} - h^i q^i)\, dy_1 \Bigg\} \ .
\end{aligned}$$

We will choose two terms on the right hand side of the above inequality and we will show that they can be estimated by $Ck/\Delta t$, where C does not depend on i,j,k,n. The other terms can be then estimated in a similar fashion.
2. Let us first recall (4.3), i.e.,

$$\begin{aligned}
&B^j(\boldsymbol{u}^j,\boldsymbol{u}^j,h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i) \\
&\equiv \int_D \left(h^j u_1^j\left(\frac{\partial\boldsymbol{u}^j}{\partial y_1} - \frac{y_2}{h^j}\frac{\partial h^j}{\partial y_1}\frac{\partial\boldsymbol{u}^j}{\partial y_2}\right) + u_2^j\frac{\partial\boldsymbol{u}^j}{\partial y_2}\right)\cdot\left(h^{i+k}\boldsymbol{u}^{i+k} - h^i\boldsymbol{u}^i\right)\, dy
\end{aligned}$$

and focus on the nonlinear term

$$\int_D u_1^j y_2\frac{\partial h^j}{\partial y_1}\frac{\partial\boldsymbol{u}^j}{\partial y_2}\cdot\left(h^{i+k}\boldsymbol{u}^{i+k}\right)\, dy$$

that we shall treat in every detail. Going back to the item 1., the corresponding sum can be estimated by

$$
\begin{aligned}
&\sum_{i=1}^{n-k}\left(\int_D\left|\sum_{j=i+1}^{i+k}|u_1^j|\left|\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right|\right|^{6/5}dy\right)^{5/6}\left(\int_D\left|\boldsymbol{u}^{i+k}\right|^6dy\right)^{1/6}c\,(\Delta t)^2\\
&\le\left[\sum_{i=1}^{n-k}\left(\int_D\left|\sum_{j=i+1}^{i+k}|u_1^j|\left|\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right|\right|^{6/5}dy\right)^{5/4}\Delta t\right]^{2/3} \qquad (5.9)\\
&\qquad\times\left[\sum_{i=1}^{n}\left(\int_D\left|\boldsymbol{u}^{i+k}\right|^6dy\right)^{1/2}\Delta t\right]^{1/3}c\Delta t\,.
\end{aligned}
$$

The first term on the right hand side of (5.9) we can further estimate using Minkowski's inequality in $L^{6/5}(D)$ and then applying Hölder inequality to get

$$
\begin{aligned}
&\left[\sum_{i=1}^{n-k}\left(\sum_{j=i+1}^{i+k}\left(\int_D|u_1^j|^{6/5}\left|\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right|^{6/5}dy\right)^{5/6}\right)^{3/2}\Delta t\right]^{2/3}\\
&\le\left[\sum_{i=1}^{n-k}\sum_{j=i+1}^{i+k}\left(\int_D|u_1^j|^{6/5}\left|\frac{\partial \boldsymbol{u}^j}{\partial y_2}\right|^{6/5}dy\right)^{5/4}k^{1/2}\Delta t\right]^{2/3} \qquad (5.10)\\
&\le\left[\sum_{i=1}^{n}\left(\int_D\left|\frac{\partial \boldsymbol{u}^i}{\partial y_2}\right|^2dy\right)^{3/4}\left(\int_D|u_1^i|^3dy\right)^{1/2}\Delta t\right]^{2/3}k\\
&\le\left[\sum_{i=1}^{n}\left(\int_D\left|\nabla\boldsymbol{u}^i\right|^2dy\right)\Delta t\right]^{1/2}\left[\sum_{i=1}^{n}\left(\int_D|u_1^i|^3dy\right)^2\Delta t\right]^{1/6}k\,.
\end{aligned}
$$

Thus, (5.9) can be estimated with the assistance of (5.10) by

$$
\|\boldsymbol{u}_n\|_{L^2(0,T;(H^1(D))^2)}\,\|\boldsymbol{u}_n\|_{L^6(0,T;(L^3(D))^2)}\,\|\boldsymbol{u}_n\|_{L^3(0,T;(L^6(D))^2)}\,ck\Delta t\,,
$$

where we have defined the step function $\boldsymbol{u}_n : [0,T] \longrightarrow \boldsymbol{V}$ for $n\in\mathbb{N}$ such that $\boldsymbol{u}_n(t)=\boldsymbol{u}^i$ for $t\in((i-1)\Delta t, i\Delta t]$, $i=0,1,\ldots,n$, $\boldsymbol{u}^0=\boldsymbol{0}$, see Section 6 below. The a priori estimates from Theorem 5.1 yield the boundedness of $\{\boldsymbol{u}_n\}_{n=1}^{\infty}$ with respect to n in the space $L^2(0,T;\boldsymbol{V})\cap L^\infty(0,T;(L^2(D))^2)$ and Proposition 3.1 gives its boundedness in the space $L^{2p/(p-2)}(0,T;(L^p(D))^2)$ for any $p\ge 2$, i.e. also for $p=3,6$.

3. As concerns boundary terms, let us estimate the nonlinear term

$$
-\frac{1}{2}\sum_{i=1}^{n-k}\sum_{j=i+1}^{i+k}\int_0^L\left|u_2^j\right|^2h^{i+k}u_2^{i+k}(y_1,1)\,dy_1(\Delta t)^2
$$

$$\leq c \sum_{i=1}^{n-k} \left(\int_0^L \left| u_2^{i+k} \right|^2 (y_1, 1) dy_1 \right)^{1/2} \left[\sum_{j=i+1}^{i+k} \left(\int_0^L \left| u_2^j \right|^4 (y_1, 1) dy_1 \right)^{1/2} \right] (\Delta t)^2$$

$$\leq c \left(\sum_{i=1}^{n} \left(\int_0^L \left| u_2^i \right|^2 (y_1, 1) dy_1 \right)^2 \Delta t \right)^{1/4}$$

$$\times \left(\sum_{i=1}^{n-k} \left[\sum_{j=i+1}^{i+k} \left(\int_0^L \left| u_2^j \right|^4 (y_1, 1) dy_1 \right)^{1/2} \right]^{4/3} \Delta t \right)^{3/4} \Delta t$$

$$\leq c \left\| \boldsymbol{u}_n \right\|_{L^4(0,T;(L^2(S))^2)} \left(\sum_{i=1}^{n-k} \sum_{j=i+1}^{i+k} \left(\int_0^L \left| u_2^j \right|^4 (y_1, 1) dy_1 \right)^{2/3} k^{1/3} \Delta t \right)^{3/4} \Delta t$$

$$\leq c \left\| \boldsymbol{u}_n \right\|_{L^4(0,T;(L^2(S))^2)} \left\| \boldsymbol{u}_n \right\|^2_{L^{8/3}(0,T;(L^4(S))^2)} k \Delta t \ .$$

In this case, the boundedness of $\{\boldsymbol{u}_n\}_{n=1}^{\infty}$ with respect to n in the space $L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D)^2)$ and Proposition 3.1 yield its boundedness in the spaces $L^{2p/(p-1)}(0,T;(L^p(S))^2)$ for any $p \geq 2$, i.e. also for $p = 2, 4$. Direct calculations with all other terms, the details of which we omit, verify (5.8). ■

Remark 5.1. Note that the test function $\boldsymbol{\omega} = h^{i+k}\boldsymbol{u}^{i+k} - h^i \boldsymbol{u}^i$ does not fulfill the discrete divergence-free condition $\text{div}_i \ \boldsymbol{\omega} = 0$ in general. This is one of the reasons why we regularize the incompressibility condition (1.1) of the original model by (2.13) and we do not work in solenoidal spaces. □

6. Existence

Let us first construct sequences of functions

$$\boldsymbol{u}_n^s \ , \ \ \boldsymbol{U}_n^s \ , \ \ \boldsymbol{U}_n \ : \ [0,T] \longrightarrow \boldsymbol{V}$$

for $n \in \mathbb{N}$ such that

$$\boldsymbol{u}_n^s(t) = \boldsymbol{u}^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{6.1}$$

$i = 0, 1, \ldots, n, \ \boldsymbol{u}^0 = \boldsymbol{0}$,

$$\boldsymbol{U}_n^s(t) = \boldsymbol{u}^i h^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{6.2}$$

$$\boldsymbol{U}_n(t) = \boldsymbol{u}^{i-1} h^{i-1} + \frac{t - (i-1)\Delta t}{\Delta t} \left(\boldsymbol{u}^i h^i - \boldsymbol{u}^{i-1} h^{i-1} \right) \tag{6.3}$$

for $t \in [(i-1)\Delta t, i\Delta t]$, $i = 1, \ldots, n$,

$$q_n^s \ , \ \ Q_n^s \ , \ \ Q_n \ : \ [0,T] \longrightarrow H^1(D)$$

$n \in \mathbb{N}$ such that

$$q_n^s(t) = q^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{6.4}$$

$i = 0, 1, \dots, n,\ q^0 = 0,$

$$Q_n^s(t) = q^i h^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{6.5}$$

$$Q_n(t) = q^{i-1}h^{i-1} + \frac{t-(i-1)\Delta t}{\Delta t}\left(q^i h^i - q^{i-1}h^{i-1}\right) \tag{6.6}$$

for $t \in [(i-1)\Delta t, i\Delta t]$, $i = 1, \dots, n$,

$$u_n^s,\ u_n\ :\ [0,T] \longrightarrow H_0^1(0,L)\ ,$$

$n \in \mathbb{N}$ such that

$$u_n^s(t) = u^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{6.7}$$

$i = 0, 1, \dots, n$,

$$u_n(t) = u^{i-1} + \frac{t-(i-1)\Delta t}{\Delta t}\left(u^i - u^{i-1}\right) \quad \text{for} \quad t \in [(i-1)\Delta t, i\Delta t]$$

and finally,

$$h_n^s,\ h_n\ :\ [0,T] \longrightarrow H_0^1(0,L)\ ,$$

$n \in \mathbb{N}$ such that

$$h_n^s(t) = h^i \qquad \text{for} \quad t \in ((i-1)\Delta t, i\Delta t], \tag{6.8}$$

$i = 0, 1, \dots, n$,

$$h_n(t) = h^{i-1} + \frac{t-(i-1)\Delta t}{\Delta t}\left(h^i - h^{i-1}\right) \quad \text{for} \quad t \in [(i-1)\Delta t, i\Delta t]\ .$$

Our plan is to pass $n \longrightarrow \infty$. Therefore, we need estimates which are uniform in n. According to the a priori estimates (5.4),(5.6) and (5.8) we observe that the sequences

$$\{\boldsymbol{u}_n\}_{n=1}^{\infty}, \{\boldsymbol{U}_n^s\}_{n=1}^{\infty}, \{\boldsymbol{U}_n\}_{n=1}^{\infty} \quad \text{are bounded in} \ \ L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;L^2(D)^2), \tag{6.9}$$

$$\|\boldsymbol{U}_n^s - \boldsymbol{U}_n\|_{L^2(0,T;(L^2(D))^2)} \le C\sqrt{\Delta t}\ , \tag{6.10}$$

$$\left\{\frac{\partial \boldsymbol{U}_n}{\partial t}\right\}_{n=1}^{\infty} \quad \text{is bounded in} \quad L^2(0,T;\boldsymbol{V}^*) + L^{4/3}(0,T;L^{4/3}(D)^2)\ , \tag{6.11}$$

$$\int_0^{T-k\Delta t}\int_D |\boldsymbol{U}_n^s(t+k\Delta t) - \boldsymbol{U}_n^s(t)|^2\, dy\, dt \le C\, k\Delta t\ , \tag{6.12}$$

$$\{\sqrt{\varepsilon}\, q_n^s\}_{n=1}^{\infty}, \{\sqrt{\varepsilon}\, Q_n^s\}_{n=1}^{\infty}, \{\sqrt{\varepsilon}\, Q_n\}_{n=1}^{\infty} \quad \text{are bounded in} \quad L^2(0,T;H^1(D)) \cap L^\infty(0,T;L^2(D))\ , \tag{6.13}$$

$$\left\|\sqrt{\varepsilon}\ (Q_n^s - Q_n)\right\|_{L^2(Q)} \le C\sqrt{\Delta t}\ , \tag{6.14}$$

$$\left\{\sqrt{\varepsilon}\frac{\partial Q_n}{\partial t}\right\}_{n=1}^{\infty} \quad \text{is bounded in} \quad L^2(0,T;H^{-1}(D)) \ , \tag{6.15}$$

$$\varepsilon \int_0^{T-k\Delta t}\int_D |Q_n^s(t+k\Delta t) - Q_n^s(t)|^2 \, dy \, dt \le C\, k\Delta t \tag{6.16}$$

and finally,

$$\{u_n^s\}_{n=1}^{\infty}, \{u_n\}_{n=1}^{\infty} \quad \text{are bounded in} \quad L^\infty(0,T;H_0^1(0,L)), \tag{6.17}$$

$$\|u_n^s - u_n\|_{L^2(0,T;L^2(0,L))} \le C\sqrt{\Delta t} \ , \tag{6.18}$$

$$\left\{\frac{\partial u_n}{\partial t}\right\}_{n=1}^{\infty} \quad \text{is bounded in} \quad L^2(0,T;L^2(0,L)) \ . \tag{6.19}$$

As a consequence we get

Lemma 6.1 *There exist a subsequence of* $\{n\}_{n=1}^{\infty}$ *and functions* $(\boldsymbol{u}, q, u) \in L^2(0,T;V) \cap L^\infty(0,T;L^2(D)^3 \times L^2(0,L))$ *(we denote this subsequence for simplicity again* $\{n\}_{n=1}^{\infty}$*), such that*

$$h_n \longrightarrow h \qquad \textit{in} \quad W^{1,\infty}(0,T;C([0,L])), \tag{6.20}$$

$$h_n^s \longrightarrow h \qquad \textit{in} \quad L^\infty(0,T;C^1([0,L])), \tag{6.21}$$

$$\left.\begin{array}{r} \boldsymbol{U}_n^s \longrightarrow h\boldsymbol{u} \\ \boldsymbol{u}_n^s \longrightarrow \boldsymbol{u} \end{array}\right\} \qquad \textit{weakly in} \quad L^2(0,T;\boldsymbol{V}), \tag{6.22}$$

strongly in $L^p(D\times(0,T))$ *for any* $1 \le p < 4$ *and in* $L^2(0,T;L^p(S))$ *for any* $p \ge 1$,

$$\begin{aligned} q_n^s &\longrightarrow q && \textit{weakly in } L^2(0,T;H^1(D)) \textit{ and} \\ & && \textit{strongly in } L^2(D\times(0,T)), && (6.23) \\ Q_n &\longrightarrow hq && \textit{weakly in } H^1(0,T;H^{-1}(D)), && (6.24) \\ u_n &\longrightarrow u && \textit{weakly in } H^1((0,L)\times(0,T)) && (6.25) \end{aligned}$$

as $n \longrightarrow \infty$.

Proof. 1. Since $h \in C^1([0,T]\times[0,L])$, (6.20) and (6.21) follow easily.
2. The weak convergence (6.22) is the consequence of (6.9) and (6.20), see (6.1), (6.2) above. To prove the strong convergence of $\boldsymbol{U}_n^s$ in L^p, choose k such that $k\Delta t < T$, $D^\delta = (\delta_1, L-\delta_1)\times(\delta_2, 1-\delta_2)$ for small $|\delta|$, $1 \ll \ell < \infty$ and

$$(\boldsymbol{U}_n^s)^\ell = \min\left\{1, \frac{\ell}{|\boldsymbol{U}_n^s|}\right\}\boldsymbol{U}_n^s.$$

The estimate (6.12) then yields

$$\int_0^{T-k\Delta t}\int_D \left|(\boldsymbol{U}_n^s)^\ell(y,t+k\Delta t) - (\boldsymbol{U}_n^s)^\ell(y,t)\right| \, dy \, dt \le C\sqrt{k\Delta t}.$$

Further, it is not difficult to show that

$$\int_0^T \int_D \left| (\boldsymbol{U}_n^s)^\ell (y+\delta,t) - (\boldsymbol{U}_n^s)^\ell (y,t) \right| \; dy \; dt \le C|\delta| \left(\|\boldsymbol{U}_n^s\|_{L^2(0,T;(H^1D)^2)} + \ell \right),$$

where, if necessary, we extend $\boldsymbol{U}_n^s$ outside of D by zero. Thus for fixed ℓ we can conclude that the set $\{(\boldsymbol{U}_n^s)^\ell\}_{n=1}^\infty$ is precompact in $L^1(D\times(0,T))^2$, see Riesz's (Kolmogorov's) compactness criteria [22, Theorem 2.13.1, p.88] or [2, Lemma 1.9]. Then using the inequality which we borrow from [2]

$$|\,(\boldsymbol{U}_n^s)^\ell - \boldsymbol{U}_n^s| \le \frac{1}{\ell}|\,(\boldsymbol{U}_n^s)\,|^2,$$

we obtain that also $\{\boldsymbol{U}_n^s\}_{n=1}^\infty$ is precompact in $L^1(D\times(0,T))^2$, therefore a subsequence exists which converges strongly in this space. Hence, we obtain strong convergence

$$\boldsymbol{U}_n^s \to h\boldsymbol{u} \quad \text{in} \quad L^1(D\times(0,T))^2$$

as $n\to\infty$. According to Proposition 3.1 and the estimate (6.9) we observe that $\boldsymbol{U}_n^s,\ \boldsymbol{u}_n^s$ are bounded in $L^4(D\times(0,T))^2$. Due to the interpolation argument for $1\le p<4$ we obtain the strong convergence of $\boldsymbol{U}_n^s$ in the space stated in second assertion of (6.22). Without a loss of generality we can consider $p\ge 2$ and on the base of Proposition 3.2 (3.26) we get

$$\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^2_{L^2(0,T;L^p(S)^2)} \le c\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^{\frac{2}{p}}_{L^2(D\times(0,T))^2}\|\boldsymbol{U}_n^s - h\boldsymbol{u}\|^{\frac{2(p-1)}{p}}_{L^2(0,T;H^1(D)^2)}.$$

Hence we obtain the third assertion of (6.22).
3. To prove (6.23), we use the estimates (6.13)–(6.14) for fixed $\varepsilon>0$ and apply the above reasoning. For the proof of (6.24) for a fixed $\varepsilon>0$ we refer to [21],[10] and we omit it here. Finally, (6.25) follows from (6.17) and (6.19). ■

Now, let us continue our consideration simply assuming that test functions $(\boldsymbol{\psi},\phi,\xi)$ in (3.21) are more regular than required, i.e. .

$$(\boldsymbol{\psi},\phi,\xi)\in C([0,T];V)\,, \qquad \boldsymbol{\psi}\in C^1(\overline{D}\times[0,T])^2\,. \tag{6.26}$$

We denote $\boldsymbol{\psi}^i(y)=\boldsymbol{\psi}(y,i\Delta t)\,,\ \phi^i(y)=\phi(y,i\Delta t)\,,\ \xi^i(y_1)=\xi(y_1,i\Delta t)$ and construct sequences of functions $\boldsymbol{\psi}_n\,,\ \boldsymbol{\psi}_n^s\,,\ \phi_n^s\,,\ \xi_n^s$ in the same fashion as above. It is straightforward to check that

$$\boldsymbol{\psi}_n \to \boldsymbol{\psi} \ \text{ in } \ H^1(D\times(0,T))\,, \quad \boldsymbol{\psi}_n^s \to \boldsymbol{\psi} \ \text{ in } \ L^\infty(0,T;C^1(\overline{D}))\,,$$
$$\phi_n^s \to \phi \ \text{ in } \ L^2(0,T;H^1(D)) \ \text{ and } \ \xi_n^s \to \xi \ \ L^\infty(0,T;H_0^1(0,L)) \ \text{ as } n\to\infty\,.$$

After these preparations it is now easy to prove the existence of a solution to our problem. Putting $(\boldsymbol{\psi}_n^s(i\Delta t),\phi_n^s(i\Delta t),\xi_n^s(i\Delta t))$ into (4.1), multiplying it by Δt and adding up through $i=1,2,\ldots,n$, after recognizable arrangements, we obtain:

$$\int_{\Delta t}^T\int_D \boldsymbol{U}_n^s(t-\Delta t)\frac{\partial\boldsymbol{\psi}_n}{\partial t}(t)\;dy\;dt \tag{6.27}$$

$$
\begin{aligned}
= \int_0^T \Bigg\{ \int_D \Bigg(& -\frac{\partial h_n}{\partial t}\frac{\partial (y_2 \boldsymbol{u}_n^s)}{\partial y_2}\cdot \boldsymbol{\psi}_n^s \\
& + \left(h_n^s u_{1n}^{\;s}\left(\frac{\partial \boldsymbol{u}_n^s}{\partial y_1} - \frac{y_2}{h_n^s}\frac{\partial h_n^s}{\partial y_1}\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}\right) + u_{2n}^{\;s}\frac{\partial \boldsymbol{u}_n^s}{\partial y_2}\right)\cdot \boldsymbol{\psi}_n^s \\
& + \frac{h_n^s}{2}\, \boldsymbol{u}_n^s\cdot \boldsymbol{\psi}_n^s \operatorname{div}_{h_n^s}\boldsymbol{u}_n^s \;-\; h_n^s\, q_n^s \operatorname{div}_{h_n^s}\boldsymbol{\psi}_n^s \Bigg) dy \;+\; \nu(\!(\boldsymbol{u}_n^s, \boldsymbol{\psi}_n^s)\!)_{h_n^s} \\
+ R\int_0^1 \Bigg(& \left(q_{out}^{n,s} - \frac{1}{2}\,|u_{1n}^{\;s}|^2\right)\psi_{1n}^{\;s}\,(L,\cdot,\cdot) - \left(q_{in}^{n,s} - \frac{1}{2}\,|u_{1n}^{\;s}|^2\right)\psi_{1n}^{\;s}\,(0,\cdot,\cdot)\Bigg) dy_2 \\
+ \int_0^L \Bigg\{ & q_w^{n,s} - \frac{1}{2}u_{2n}^{\;s}\left(u_{2n}^{\;s} - \frac{\partial h_n}{\partial t}\right) + \kappa\left(u_{2n}^{\;s} - \lambda u_n^s - (1-\lambda)\frac{\partial h_n}{\partial t}\right)\Bigg\}\,\psi_{2n}^{\;s}\, dy_1 \\
+ \varepsilon \left\langle \frac{\partial Q_n}{\partial t}, \phi_n^s\right\rangle & - \int_D \left(\varepsilon\frac{\partial h_n}{\partial t}\frac{\partial (y_2 q_n^s)}{\partial y_2}\phi_n^s \;+\; h_n^s \operatorname{div}_{h_n^s}\boldsymbol{u}_n^s\;\phi_n^s\right) dy \\
+ \varepsilon(\!(q_n^s, \phi_n^s)\!)_{h_n^s} & + \frac{\varepsilon}{2}\int_0^L \frac{\partial h_n}{\partial t}(y_1,t)\; q_n^s \phi_n^s\;(y_1,1,t)\, dy_1 \\
+ \int_0^L \Bigg\{ & \frac{\partial u_n}{\partial t}\xi_n^s + c\,\frac{\partial u_n^s}{\partial y_1}\frac{\partial \xi_n^s}{\partial y_1} + a\frac{\partial}{\partial y_1}\int_0^t u_n^s(y_1,s)ds\,\frac{\partial \xi_n^s}{\partial y_1} \\
+ b\int_0^t & u_n^s(y_1,s)ds\;\xi_n^s + \frac{\kappa}{E}\left(\lambda u_n^s + (1-\lambda)\frac{\partial h_n}{\partial t} - u_{2n}^{\;s}\right)\xi_n^s\Bigg\}\,(y_1,t)\,dy_1\Bigg\}\, dt\,,
\end{aligned}
$$

where $q_{\dots}^{n,s}(\cdot,t) = g_{\dots}^i(\cdot)$ for $t \in ((i-1)\Delta t, i\Delta t)$, see (4.1). Passing to limits as $n \longrightarrow \infty$ we deduce that $(\boldsymbol{u}, q, u)$ satisfy (3.21) for test functions $(\boldsymbol{\psi}, \phi, \xi)$ with the stated regularity (6.26), where, however, the leading term

$$
\int_0^T \left\langle \frac{\partial (h\boldsymbol{u})}{\partial t}, \boldsymbol{\psi}\right\rangle dt \quad \text{is replaced by} \quad -\int_0^T\int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t}\; dydt\;.
$$

Due to the approximation argument we can conclude that (3.21) hold for every $(\boldsymbol{\psi}, \phi, \xi) \in L^2(0,T;V), \boldsymbol{\psi} \in L^4(0,T;L^4(D)^2)$.

It remains to show the existence of the time derivative of $h\boldsymbol{u}$. We have just proved that there exists

$$
\mathcal{L} \in \left(L^4(0,T;\boldsymbol{V})\right)^* \;= L^{4/3}(0,T;\boldsymbol{V}^*)
$$

such that

$$
-\int_0^T\int_D h\boldsymbol{u}\frac{\partial \boldsymbol{\psi}}{\partial t}\; dydt \;=\; \int_0^T \langle \mathcal{L}(t), \boldsymbol{\psi}(t)\rangle_V\, dt \tag{6.28}
$$

for any $\boldsymbol{\psi} \in L^4(0,T;\boldsymbol{V}) \cap H^{1,1}(0,T;L^2(D)^2),\ \boldsymbol{\psi}(T) = 0$, where $\langle\cdot,\cdot\rangle_V$ denotes the pairing between $\boldsymbol{V}^*$ and $\boldsymbol{V}$. Putting

$$
\int_0^T \langle h\boldsymbol{u}, \boldsymbol{\varpi}\rangle_V\, dt = \int_0^T\int_D h\boldsymbol{u}\cdot\boldsymbol{\varpi}\; dydt \qquad \text{for } \boldsymbol{\varpi} \in L^1(0,T;\boldsymbol{V})\;,
$$

(6.28) easily yields

$$
-\int_0^T \langle h\boldsymbol{u}(t), \boldsymbol{w}\rangle_V\, \xi'(t)\; dt = \int_0^T \langle \mathcal{L}(t), \boldsymbol{w}\rangle_V\, \xi(t)\; dt \qquad \forall \boldsymbol{w} \in \boldsymbol{V}
$$

and for all $\xi \in C_0^\infty(0,T)$, i.e.

$$\mathcal{L} \;=\; \frac{\partial(h\boldsymbol{u})}{\partial t} \;\in\; L^{4/3}(0,T;\boldsymbol{V}^*)\,. \tag{6.29}$$

Moreover, it is not difficult to see that

$$-\int_0^T\int_D h\boldsymbol{u}\frac{\partial\boldsymbol{\psi}}{\partial t}\,dydt \;=\; \int_0^T\Big(\langle\mathcal{F}(t),\boldsymbol{\psi}(t)\rangle_V \;+\; \int_D \mathcal{N}(t)\cdot\boldsymbol{\psi}(t)\,dy\Big)\,dt$$

for all $\boldsymbol{\psi} \in X \equiv L^2(0,T;\boldsymbol{V}) \cap L^4(0,T;L^4(D)^2)$ and for given $\mathcal{F} \in L^2(0,T;V^*)$ and $\mathcal{N} \in L^{4/3}(0,T;L^{4/3}(D)^2)$. Therefore, (6.28) and (6.29) easily yield

$$\int_0^T\Big\langle\frac{\partial(h\boldsymbol{u})}{\partial t},\boldsymbol{\psi}\Big\rangle_V\,dt \;=\; \int_0^T\Big(\langle\mathcal{F}(t),\boldsymbol{\psi}(t)\rangle_V \;+\; \int_D \mathcal{N}(t)\cdot\boldsymbol{\psi}(t)\,dy\Big)\,dt \tag{6.30}$$

for any $\boldsymbol{\psi} \in L^4(0,T;\boldsymbol{V})$. As the right hand side of (6.30) is well defined for any $\boldsymbol{\psi} \in X$, we define

$$\int_0^T\Big\langle\frac{\partial(h\boldsymbol{u})}{\partial t},\boldsymbol{\psi}\Big\rangle\,dt \;=\; \int_0^T\Big(\langle\mathcal{F}(t),\boldsymbol{\psi}(t)\rangle_V \;+\; \int_D \mathcal{N}(t)\cdot\boldsymbol{\psi}(t)\,dy\Big)\,dt \tag{6.31}$$

for any $\boldsymbol{\psi} \in X$. Consequently

$$\frac{\partial(h\boldsymbol{u})}{\partial t} \in X^* = L^2(0,T;\boldsymbol{V}^*) + L^{4/3}(0,T;L^{4/3}(D)^2), \tag{6.32}$$

see [21], [23].

Lemma 6.2 *Assume that the item 1) of Definition 3.1 is satisfied. Then for almost all t the following formula holds*

$$\frac{1}{2}\int_D |\boldsymbol{u}|^2(t)h(t)\,dy + \frac{1}{2}\int_0^t\int_D |\boldsymbol{u}|^2\,\frac{\partial h}{\partial t}\,dy\,ds = \int_0^t\Big\langle\frac{\partial(\boldsymbol{u}h)}{\partial t},\boldsymbol{u}\Big\rangle\,ds \tag{6.33}$$

and for q holds the same.

Proof of Lemma 6.2. 1. Note first that if $A, B \in \mathbb{R}$ and $a, b \in \mathbb{R}_+$, then

$$A^2a - B^2b = 2(Aa - Bb)A - AB(a-b) - (A\sqrt{a} - B\sqrt{b})^2 + AB(\sqrt{a} - \sqrt{b})^2$$

and

$$A^2a - B^2b = 2(Aa - Bb)B - AB(a-b) + (A\sqrt{a} - B\sqrt{b})^2 - AB(\sqrt{a} - \sqrt{b})^2,$$

(Alt's-Luckhaus' idea from [2, Lemma 1.5]). Therefore we have for almost all $t > 0$ point-wise in D

$$\begin{aligned}&|\boldsymbol{u}|^2\,h\;(t) - |\boldsymbol{u}|^2\,h\;(t-\Delta t) \\ &\le 2\,[\boldsymbol{u}h\;(t) - \boldsymbol{u}h\;(t-\Delta t)]\cdot\boldsymbol{u}(t) - \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\,[h(t) - h(t-\Delta t)] \\ &\quad+\;\boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\sqrt{h(t)} - \sqrt{h(t-\Delta t)}\right]^2\end{aligned} \tag{6.34}$$

and

$$|\boldsymbol{u}|^2\, h\,(t) - |\boldsymbol{u}|^2\, h\,(t-\Delta t) \tag{6.35}$$
$$\geq 2\,[\boldsymbol{u}h\,(t) - \boldsymbol{u}h\,(t-\Delta t)]\cdot \boldsymbol{u}(t-\Delta t) - \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\,[h(t)-h(t-\Delta t)]$$
$$-\ \boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\sqrt{h(t)}-\sqrt{h(t-\Delta t)}\right]^2\ ,$$

where $\boldsymbol{u}(t) = \boldsymbol{0}$ for $-\Delta t \leq t \leq 0$.
2. Next we integrate the inequalities (6.34) and (6.35) over $D\times(0,\tau)$ and divide by Δt. The first two terms on the right hand sides equal

$$2\int_0^\tau\int_D\left[\frac{\boldsymbol{u}h\,(t)-\boldsymbol{u}h\,(t-\Delta t)}{\Delta t}\right]\cdot\boldsymbol{u}(t)\,dy\,dt$$
$$-\int_0^\tau\int_D\boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\frac{h(t)-h(t-\Delta t)}{\Delta t}\right]\,dy\,dt$$
$$=-2\int_0^\tau\left\langle\frac{\partial(\boldsymbol{u}h)}{\partial t},[\boldsymbol{u}]_{\Delta t}\right\rangle dt + 2\int_D[\boldsymbol{u}]_{\Delta t}(\tau)\cdot\boldsymbol{u}(\tau)\,h(\tau)\,dy$$
$$+\int_0^\tau\int_D\boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\frac{h(t)-h(t-\Delta t)}{\Delta t}\right]\,dy\,dt$$

and we arrive at

$$\begin{aligned}\left[|\boldsymbol{u}|^2h\right]_{\Delta t}(\tau) \quad\leq\quad & -2\int_0^\tau\left\langle\frac{\partial(\boldsymbol{u}h)}{\partial t},[\boldsymbol{u}]_{\Delta t}\right\rangle dt\\ & +2\int_D[\boldsymbol{u}]_{\Delta t}(\tau)\cdot\boldsymbol{u}(\tau)\,h(\tau)\,dy\\ & +\int_0^\tau\int_D\boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left[\frac{h(t)-h(t-\Delta t)}{\Delta t}\right]\,dy\,dt\\ & -\int_0^\tau\int_D\boldsymbol{u}(t)\cdot\boldsymbol{u}(t-\Delta t)\left(\frac{\sqrt{h(t)}-\sqrt{h(t-\Delta t}}{\Delta t}\right)^2\Delta t\,dy\,dt\end{aligned}$$

where we have denoted

$$[\boldsymbol{u}]_{\Delta t}(t) \equiv \frac{1}{\Delta t}\int_{t-\Delta t}^t \boldsymbol{u}(s)\,ds\ . \tag{6.36}$$

The reverse inequality we get by the same fashion.
3. Finally, let $\Delta t \longrightarrow 0$ in the above inequality. We get

$$\int_D|\boldsymbol{u}|^2(\tau)h(\tau)\,dy$$
$$\leq -2\int_0^\tau\left\langle\frac{\partial(\boldsymbol{u}h)}{\partial t},\boldsymbol{u}\right\rangle dt + 2\int_D|\boldsymbol{u}|^2(\tau)\,h(\tau)\,dy + \int_0^\tau\int_D|\boldsymbol{u}|^2\frac{\partial h}{\partial t}\,dy\,dt\ ,$$

that with the reverse inequality easily imply (6.33). ■

The following theorem summarizes the main result of this section.

Theorem 6.3 *Assume $\varepsilon > 0$ and $\kappa > 0$ are given. Let h and q_{in}, q_{out} and q_w satisfy (1.2) and* (3.19). *Then there exists a unique solution $(\boldsymbol{u}_\varepsilon, q_\varepsilon, u_\varepsilon)$ of Problem (3.6)-(3.16) in the sense of Definition* 3.1 *such that (6.33) holds.*

Remark 6.1. The uniqueness follows from the result of the next section.
For $\lambda = 1$ one can see using an approximation argument that the assertion of the theorem is valid for $h \in W^{1,\infty}(\mathcal{S}_T)$, $\mathcal{S}_T = (0,L) \times (0,T)$, without higher regularity assumptions (3.19). □

7. Uniqueness, continuous dependence on data

The main aim of this section is to show a continuous dependence of solutions of Problem (3.6)–(3.16) on the data h, q_{in}, q_w and q_{out}.

Theorem 7.1 *Let $(\boldsymbol{u}^1, q^1, u^1)$ and $(\boldsymbol{u}^2, q^2, u^2)$ be weak solutions of the initial boundary value problem (3.6)–(3.16) in the sense of Definition 3.1 with given functions h^1, q^1_{in}, q^1_w, q^1_{out} and h^2, q^2_{in}, q^2_w, q^2_{out}, respectively, and suppose that*

$$0 < \alpha \le h^j(y_1,t) \le \alpha^{-1}\ , \quad \left|\frac{\partial h^j}{\partial y_1}(y_1,t)\right| + \left|\frac{\partial h^j}{\partial t}(y_1,t)\right| \le K \tag{7.1}$$

for given α, K and for all $(y_1,t) \in [0,L] \times [0,T]$, $j = 1,2$.
Then for almost all $t \in [0,T]$

$$\begin{aligned}
&\int_D \left|h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2\right|^2(t)\,dy + \nu\int_0^t\int_D \left|\nabla\left(h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2\right)\right|^2 dy\,ds\\
&+\varepsilon\left(\int_D \left|h^1q^1 - h^2q^2\right|^2(t)\,dy + \int_0^t\int_D \left|\nabla\left(h^1q^1 - h^2q^2\right)\right|^2 dy\,ds\right)\\
&+\int_0^L \left|u^1-u^2\right|^2(t)\,dy_1 + c\int_0^t\int_0^L \left|\frac{\partial(u^1-u^2)}{\partial y_1}\right|^2 dy_1\,ds\\
&+a\int_0^L\left(\int_0^t\left(\frac{\partial(u^1-u^2)}{\partial y_1}\right)ds\right)^2 dy_1 + b\int_0^L\left(\int_0^t (u^1-u^2)\,ds\right)^2 dy_1\\
&\le\ \|h^1-h^2\|^2_{W^{1,\infty}(\mathcal{S}_t)}\,\omega(t)\ +\ C\ \left\|q^1_{\scriptscriptstyle S} - q^2_{\scriptscriptstyle S}\right\|^2_{L^2(S_t)}\ ,
\end{aligned} \tag{7.2}$$

where $\omega(t) \downarrow 0$ as $t \to 0$, $C > 0$ and

$$\|q_{\scriptscriptstyle S}\|^2_{L^2(S_t)} \equiv \int_0^t\left(\|q_{out}\|^2_{L^2(S_{out})} + \|q_{in}\|^2_{L^2(S_{in})} + \|q_w\|^2_{L^2(S_w)}\right)ds.$$

Proof. 1. Note first that

$$\boldsymbol{\zeta} = (\boldsymbol{\psi}, \phi, E\xi) \equiv (h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2, h^1q^1 - h^2q^2, E(u^1-u^2)) \tag{7.3}$$

is the admissible test function in the weak formulation of our problem for both $\boldsymbol{w}^1 = (\boldsymbol{u}^1, q^1, u^1)$ and $\boldsymbol{w}^2 = (\boldsymbol{u}^2, q^2, u^2)$, see (3.21). Therefore, we can substract both identities. Performing tedious but straightforward manipulations, with the assistance of *Lemma 6.2*, we get

$$\begin{aligned}
&\frac{1}{2}\int_D \left(|\boldsymbol{\psi}|^2+\varepsilon|\phi|^2\right)(t)\,dy+\frac{E}{2}\int_0^L|\xi|^2(t)\,dy_1\\
&+\int_0^t\left\{\nu a(\psi_1,\psi_1)+\nu a(\psi_2,\psi_2)+\varepsilon a(\phi,\phi)+Ec\|\xi\|^2_{H_0^1(0,L)}\right\}ds \qquad (7.4)\\
&+\frac{Ea}{2}\int_0^L\left(\int_0^t\frac{\partial\xi}{\partial y_1}\,ds\right)^2dy_1+\frac{Eb}{2}\int_0^L\left(\int_0^t\xi\,ds\right)^2dy_1\\
&=-\int_0^t\left(b^1(\boldsymbol{w}^1,\boldsymbol{w}^1,\boldsymbol{\zeta})-b^2(\boldsymbol{w}^2,\boldsymbol{w}^2,\boldsymbol{\zeta})\right)ds\\
&-\kappa\int_0^t\int_0^L(\xi-\psi_2)\left(\lambda\xi+(1-\lambda)\left(\frac{\partial h^1}{\partial t}-\frac{\partial h^2}{\partial t}\right)-(u_2^1-u_2^2)\right)dy_1\,ds\\
&-R\int_0^t\int_0^1\left((q^1_{out}-q^2_{out})\psi_1(L,y_2,s)-(q^1_{in}-q^2_{in})\psi_1(0,y_2,s)\right)dy_2\,ds\\
&-\int_0^t\int_0^L(q^1_w-q^2_w)\psi_2(y_1,1,s)\,dy_1ds\\
&+\int_0^t\int_D\left[h^1q^1 div_{h^1}\boldsymbol{\psi}-h^2q^2 div_{h^2}\boldsymbol{\psi}-\left(h^1\,div_{h^1}\boldsymbol{u}^1-h^2\,div_{h^2}\boldsymbol{u}^2\right)\phi\right]dy\,ds\\
&-\int_0^t\left\{\frac{1}{2}\int_0^L\left[\frac{1}{h^1}\frac{\partial h^1}{\partial t}-\frac{1}{h^2}\frac{\partial h^2}{\partial t}\right]h^2(u_2^2\psi_2+\varepsilon q^2\phi)dy_1\right.\\
&\qquad+\frac{1}{2}\int_0^L\frac{1}{h^1}\frac{\partial h^1}{\partial t}\left((\psi_2)^2+\varepsilon\phi^2\right)dy_1\\
&\qquad-\int_D\frac{1}{h^1}\frac{\partial h^1}{\partial t}\left(\frac{\partial\,(y_2\boldsymbol{\psi})}{\partial y_2}\cdot\boldsymbol{\psi}+\varepsilon\frac{\partial\,(y_2\phi)}{\partial y_2}\phi\right)dy\\
&\qquad\left.-\int_D\left[\frac{1}{h^1}\frac{\partial h^1}{\partial t}-\frac{1}{h^2}\frac{\partial h^2}{\partial t}\right]\left(\frac{\partial\left(y_2\boldsymbol{u}^2\right)}{\partial y_2}\cdot\boldsymbol{\psi}+\varepsilon\frac{\partial\left(y_2q^2\right)}{\partial y_2}\phi\right)h^2dy\right\}ds\\
&+\nu\Big(\mathcal{R}(\psi_1)\;+\;\mathcal{R}(\psi_2)\Big)\;+\;\varepsilon\mathcal{R}(\phi)\;,
\end{aligned}$$

where

$$a(\phi,\phi)=\int_D\left(\left(\frac{\partial\phi}{\partial y_1}-\frac{y_2}{h^2}\frac{\partial h^2}{\partial y_2}\frac{\partial\phi}{\partial y_2}\right)^2+\frac{1}{(h^2)^2}\left(\frac{\partial\phi}{\partial y_2}\right)^2\right)dy$$

and

$$\begin{aligned}
\mathcal{R}(\phi)\quad=\quad&\int_0^t\int_D\left(\frac{1}{2}\frac{\partial(\phi)^2}{\partial y_1}\frac{1}{h^2}\frac{\partial h^2}{\partial y_1}+\frac{1}{2}\frac{\partial(\phi)^2}{\partial y_2}\frac{y_2}{(h^2)^2}\left(\frac{\partial h^2}{\partial y_1}\right)^2\right. \qquad (7.5)\\
&+\left[\frac{1}{h^1}\frac{\partial h^1}{\partial y_1}-\frac{1}{h^2}\frac{\partial h^2}{\partial y_1}\right]\left(\frac{\partial\phi}{\partial y_1}\frac{\partial(y_2h^1q^1)}{\partial y_2}+\frac{\partial\phi}{\partial y_2}\frac{\partial(y_2h^1q^1)}{\partial y_1}\right)\\
&-\left[\frac{1}{(h^1)^2}-\frac{1}{(h^2)^2}\right]\frac{\partial\phi}{\partial y_2}\frac{\partial(h^1q^1)}{\partial y_2}
\end{aligned}$$

$$-\left[\left(\frac{1}{h^1}\frac{\partial h^1}{\partial y_1}\right)^2-\left(\frac{1}{h^2}\frac{\partial h^2}{\partial y_1}\right)^2\right]\frac{\partial \phi}{\partial y_2}y_2\left(h^1q^1+y_2\frac{\partial(h^1q^1)}{\partial y_2}\right)\Big)dy\,ds\ .$$

2. To estimate the first term on the right-hand side of (7.4), let us recall (4.3) and (4.12). Then it is not difficult but a little tedious to verify that

$$\begin{aligned} b^1(\boldsymbol{w}^1,\boldsymbol{w}^1,\boldsymbol{\zeta})-b^2(\boldsymbol{w}^2,\boldsymbol{w}^2,\boldsymbol{\zeta}) = \\ &+\frac{1}{2}B^2(\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{u}^2,\boldsymbol{\psi})-\frac{1}{2}B^2(\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{\psi},\boldsymbol{u}^2)\\ &+\frac{1}{2}B^1(\boldsymbol{u}^1,\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{\psi})-\frac{1}{2}B^1(\boldsymbol{u}^1,\boldsymbol{\psi},\boldsymbol{u}^1-\boldsymbol{u}^2)\\ &+\frac{1}{2}\int_D\Big\{u_1^1\left[h^1-h^2\right]\Big(\frac{\partial \boldsymbol{u}^2}{\partial y_1}\cdot\boldsymbol{\psi}-\boldsymbol{u}^2\cdot\frac{\partial\boldsymbol{\psi}}{\partial y_1}\Big)\\ &\qquad -y_2\,u_1^1\left[\frac{\partial h^1}{\partial y_1}-\frac{\partial h^2}{\partial y_1}\right]\Big(\frac{\partial \boldsymbol{u}^2}{\partial y_2}\cdot\boldsymbol{\psi}-\boldsymbol{u}^2\cdot\frac{\partial\boldsymbol{\psi}}{\partial y_2}\Big)\Big\}dy. \end{aligned} \tag{7.6}$$

Recalling (3.9), after some manipulation we also obtain

$$\begin{aligned} &h^1q^1\mathrm{div}_{h^1}\boldsymbol{\psi}-h^2q^2\mathrm{div}_{h^2}\boldsymbol{\psi}-\left(h^1\phi\ \mathrm{div}_{h^1}\boldsymbol{u}^1-h^2\phi\ \mathrm{div}_{h^2}\boldsymbol{u}^2\right)=\\ &-y_2\left[\frac{1}{h^1}\frac{\partial h^1}{\partial y_1}\frac{1}{h^2}\frac{\partial h^2}{\partial y_1}\right]\left(h^1q^1\frac{\partial\psi_1}{\partial y^2}+h^2\frac{\partial u_1^1}{\partial y^2}\phi\right)-\left[\frac{h^1-h^2}{h^2}\right]\left(q^1\frac{\partial\psi_2}{\partial y^2}+\frac{\partial u_2^1}{\partial y^2}\phi\right)\\ &-\left(\frac{\partial h^1}{\partial y_1}\frac{1}{h_2}\psi_1+\frac{\partial h^1}{\partial y_1}u_1^1\left[\frac{h^1-h^2}{h^2}\right]+u_1^2\left[\frac{\partial h^1}{\partial y_1}-\frac{\partial h^2}{\partial y_1}\right]\right)\phi\ . \end{aligned}$$

3. In the sequel, we attempt to estimate the right-hand side of (7.4) in order to get the following differential inequality

$$g(t)\le\vartheta(t)+\int_0^t\iota(s)g(s)\ ds \qquad \forall t\in[0,T] \tag{7.7}$$

for

$$g(t)\equiv\frac{1}{2}\int_D\left(|\boldsymbol{\psi}|^2+\varepsilon|\phi|^2\right)(t)\,dy+\frac{E}{2}\int_0^L|\xi|^2(t)\,dy_1$$

and some continuous function ϑ and integrable function ι on $[0,T]$, $\vartheta,\iota\ge 0$. Gronwall's Lemma (see e.g. [10, Lemma 8.2.29]) then yields

$$g(t)\le\vartheta(t)+\int_0^t\vartheta(s)\iota(s)e^{\int_s^t\iota(\tau)d\tau}ds \qquad t\in[0,T]. \tag{7.8}$$

4. Let us begin with the first term on the right-hand side of (7.6), i.e.

$$\begin{aligned} &B^2(\boldsymbol{u}^1-\boldsymbol{u}^2,\boldsymbol{u}^2,\boldsymbol{\psi})=\\ &\int_D\left(h^2(u_1^1-u_1^2)\left(\frac{\partial\boldsymbol{u}_1^2}{\partial y_1}-\frac{y_2}{h^2}\frac{\partial h^2}{\partial y_1}\frac{\boldsymbol{u}^2}{\partial y_2}\right)+(u_2^1-u_2^2)\frac{\partial\boldsymbol{u}^2}{\partial y_2}\right)\cdot\boldsymbol{\psi}\ dy \end{aligned}$$

and focus on its first term

$$\begin{aligned}
&\int_D h^2(u_1^1-u_1^2)\frac{\partial \boldsymbol{u}^2}{\partial y_1}\cdot\boldsymbol{\psi}\;dy\\
&=\int_D \frac{h^2}{h^1}\left((h^1-h^2)u_1^1+\psi_1\right)\frac{\partial \boldsymbol{u}^2}{\partial y_1}\cdot\boldsymbol{\psi}\;dy\\
&\le \left|\frac{h^2}{h^1}\right|_\infty \|\nabla\boldsymbol{u}^2\|_{L^2}\|\boldsymbol{\psi}\|_{L^4}\Big(\left|h^1-h^2\right|_\infty\|\boldsymbol{u}^1\|_{L^4}+\|\boldsymbol{\psi}\|_{L^4}\Big)\\
&\le C_1\left|h^2-h^1\right|_\infty^2\|\nabla\boldsymbol{u}^2\|_{L^2}^2+\mu_1\|\nabla\boldsymbol{\psi}\|_{L^2}^2\\
&\qquad\qquad+C_2(\mu_1)\|\boldsymbol{\psi}\|_{L^2}^2\left(\|\nabla\boldsymbol{u}^1\|_{L^2}^2+\|\nabla\boldsymbol{u}^2\|_{L^2}^2\right)\;,
\end{aligned}$$

for $0<\mu_1$ to be determined later. Here Proposition 3.1 has been applied. We omit similar computations which estimate the remaining terms of the right-hand side of (7.4) and note that

$$a(\phi,\phi)\ge\frac{\alpha^2}{2+K^2}\int_D|\nabla\phi|^2\,dy\qquad\forall\phi\in H^1(D),$$

see Lemma 4.3. Hence, we arrive at

$$\begin{aligned}
g(t)\qquad &+\nu\left(\frac{\alpha^2}{2+K^2}-\mu_1\right)\int_0^t\|\nabla\boldsymbol{\psi}\|_{L^2}^2ds+\varepsilon\left(\frac{\alpha^2}{2+K^2}-\mu_2\right)\int_0^t\|\nabla\phi\|_{L^2}^2ds\\
&+\frac{a}{2}\int_0^L\left(\int_0^t\frac{\partial\xi}{\partial y^1}ds\right)^2dy_1+\frac{b}{2}\int_0^L\left(\int_0^t\xi\,ds\right)^2dy_1\\
&+\kappa\int_0^t\Big(\lambda\|\xi\|_{L^2(0,L)}^2+\alpha\|\psi_2\|_{L^2(0,L)}^2\Big)\,ds+c\int_0^t\|\nabla\xi\|_{L^2(0,L)}^2ds\\
&\le\;\vartheta(t)+\int_0^t\iota(s)g(s)\,ds,
\end{aligned}\tag{7.9}$$

where

$$\vartheta(t)\equiv C_3\,\|h^1-h^2\|_{W^{1,\infty}(\mathcal{S}_t)}^2\tilde\omega(t)\;+\;C_4\,\left\|q_S^1-q_S^2\right\|_{L^2(S_t)}^2\;,$$

where

$$\tilde\omega(t)\equiv\int_0^t\left(\|\nabla\boldsymbol{u}^2\|_{L^2}^2+\varepsilon\|\nabla q^2\|_{L^2}^2+1\right)\,ds\tag{7.10}$$

and

$$\iota(t)\equiv C_5(\kappa,\alpha,K,\mu_1,\mu_2)\Big(\|\nabla\boldsymbol{u}^1\|_{L^2}^2(t)+\|\nabla\boldsymbol{u}^2\|_{L^2}^2(t)+1\Big),$$

$C_3=C_3(\alpha,K,\kappa)$. Note that $\vartheta(t)$ is non-decreasing.
5. After choosing $\mu_1=\mu_2=\alpha^2/(4+2K^2)$ and omitting the positive terms on the left-hand side of (7.9) except of $g(t)$ and applying Gronwall's lemma (7.8), we can estimate

$$g(t)\le\vartheta(t)\left[1+\int_0^T\iota(\tau)\;d\tau e^{\int_0^T\iota(\tau)\;d\tau}\right]\equiv\vartheta(t)C_6.\tag{7.11}$$

Finally, we estimate $g(s)$ at the right-hand side of (7.9) with the assistance of (7.11). The required estimate (7.2) follows now easily if we take the estimates (7.9) and (7.11) into account. Note that $\omega(t) \equiv \mathcal{C}\tilde{\omega}(t)$ for some positive constant $\mathcal{C}$. ■

Remark 7.1. Let us note that the test function used in the previous proof, $\boldsymbol{\psi} = h^1\boldsymbol{u}^1 - h^2\boldsymbol{u}^2$ does not fulfill the divergence-free condition in general. We overcome this difficulty by the regularization of the incompressibility condition (1.1) by (2.13). □

8. Problem with $\varepsilon = 0$

The final goal of this section is two-fold, first we let

$$\varepsilon \longrightarrow 0 \qquad \text{and then} \qquad \kappa^{-1} = \varepsilon \longrightarrow 0\,.$$

We will show that by the same procedure as before for subsequences $\boldsymbol{u}_\varepsilon$, u_ε converge strongly to a weak solution $\boldsymbol{u}, u$. First we obtain an a priori estimate by testing (3.21) with $(\boldsymbol{u}_\varepsilon, q_\varepsilon, \lambda E u_\varepsilon + E(1-\lambda)\partial_t h)$. Then we control the time dependence by multiplying with time differences

$$\left(\partial_t^\tau(h\boldsymbol{u}_\varepsilon)\,,\ \partial_t^\tau(hq_\varepsilon)\,,\ \partial_t^\tau\left(\lambda h u_\varepsilon + (1-\lambda)h\frac{\partial h}{\partial t}\right)\right),$$

where we denote $\partial_t^\tau f \equiv f(t+\tau) - f(t)$, for $\tau > 0$, which result in the compactness of the functions $(\boldsymbol{u}_\varepsilon\,, \sqrt{\varepsilon}q_\varepsilon\,, u_\varepsilon)$ in L^2.
1. Inserting $(\boldsymbol{\psi}, \phi, \xi) = (\boldsymbol{u}_\varepsilon, q_\varepsilon, E\lambda u_\varepsilon + E(1-\lambda)\partial_t h)$ into (3.21) with T replaced by t we obtain by Lemma 6.2 and Gronwall's lemma the a priori estimate

$$\begin{aligned}
&\int_D h(t)\left(|\boldsymbol{u}_\varepsilon|^2 + \varepsilon|q_\varepsilon|^2\right)(t)\,dy + \frac{\lambda E}{2}\int_0^L |u_\varepsilon(t)|^2\,dy_1 \qquad (8.1)\\
&+\frac{\alpha}{2+K^2}\int_0^t\int_D \left(\nu\,|\nabla\boldsymbol{u}_\varepsilon|^2 + \varepsilon\,|\nabla q_\varepsilon|^2\right)dy\,ds\\
&+2\kappa\int_0^t\int_0^L\left|u_{2\varepsilon} - \left(\lambda u_\varepsilon + (1-\lambda)\frac{\partial h}{\partial t}\right)\right|^2 dy_1\,ds\\
&+\frac{\lambda E}{2}\left[c\int_0^t\int_0^L\left|\frac{\partial u_\varepsilon}{\partial y_1}\right|^2 dy_1\,ds\right.\\
&\qquad\left. + a\int_0^L\left|\int_0^t\frac{\partial u_\varepsilon}{\partial y_1}\,ds\right|^2 dy_1 + b\int_0^L\left|\int_0^t u_\varepsilon\,ds\right|^2 dy_1\right] \le\ \mathcal{R}(t)
\end{aligned}$$

for a.e. $t \in (0,T)$, where

$$\mathcal{R}(t) \equiv \left\{\frac{2E(1-\lambda)^2}{\lambda}\max_{0\le\tau\le t}\left[\int_0^L\left(\left|\frac{\partial h}{\partial t}\right|^2 + Ea\left|\frac{\partial h}{\partial y_1}\right|^2 + Eb\,|h|^2\right)(\tau)dy_1\right]\right.$$

$$+\ C\,\alpha^{-1}\left(2+K^2\right)\|q_S\|^2_{L^2(S_t)}$$
$$+\frac{4(1-\lambda)^2E}{\lambda c}\int_0^t\int_0^L\left(\left|\frac{\partial^2 h}{\partial y_1\partial t}\right|^2+\left|\frac{\partial^2 h}{\partial t^2}\right|^2+a\left|\frac{\partial h}{\partial y_1}\right|^2+b\,|h|^2\right)dy_1ds\Bigg\}\,e^{Mt/\alpha}$$

which proves that for a subsequence $(\boldsymbol{u}_\varepsilon,\sqrt{\varepsilon}q_\varepsilon,u_\varepsilon)$ has a weak limit $(\boldsymbol{u},\vartheta,u)$ in $L^2(0,T;V)$ and that

$$\mathrm{div}_h\,\boldsymbol{u}=0 \qquad\qquad \text{a.e. on}\quad D\times(0,T)\,. \tag{8.2}$$

To see (8.2) let us insert $(\boldsymbol{\psi},\phi,\xi)=(\mathbf{0},\phi,0)$ for sufficiently smooth ϕ into (3.21) to get

$$\begin{aligned}&\int_0^T\int_D h\,\phi\,\mathrm{div}_h\boldsymbol{u}_\varepsilon\,dy\,dt \qquad (8.3)\\ &=\varepsilon\int_0^T\left\{\int_D\left(hq_\varepsilon\,\frac{\partial\phi}{\partial t}+\frac{\partial h}{\partial t}\frac{\partial(y_2q_\varepsilon)}{\partial y_2}\phi\right)dy\ -\ ((q_\varepsilon,\phi))_h\right\}dt\\ &-\frac{\varepsilon}{2}\int_0^T\int_0^L\frac{\partial h}{\partial t}(y_1,t)\,q_\varepsilon\phi\,(y_1,1,t)\,dy_1dt\,.\end{aligned}$$

Due to the estimate (8.1) concerning q_ε one can see that the right hand side of (8.3) tends to zero as $\varepsilon\longrightarrow 0$ and we arrive at (8.2).
2. The second step in proving the compactness is to show that

$$\begin{aligned}&\int_0^{T-\tau}\int_D\left(|(h\boldsymbol{u}_\varepsilon)(t+\tau)-(h\boldsymbol{u}_\varepsilon)(t)|^2+\varepsilon\,|(hq_\varepsilon)(t+\tau)-(hq_\varepsilon)(t)|^2\right)dy\,dt\\ &+\int_0^{T-\tau}\int_0^L|u_\varepsilon(t+\tau)-u_\varepsilon(t)|^2\,dy_1\,dt\ \le\ C\,\tau\end{aligned} \tag{8.4}$$

for a positive constant C independent on ε,κ and τ. (8.4) can be obtained in the following way. Inserting $\chi^\delta_{(t,t+\tau)}(\boldsymbol{w},p,Ev)$, $(\boldsymbol{w},p,v)\in V$ as a test function into (3.21), where $\chi^\delta_{(t,t+\tau)}$ is a smooth approximation of the characteristic function of the interval $(t,t+\tau)$, we obtain after letting $\delta\to 0$:

$$\int_D\left(\partial_t^\tau(h\boldsymbol{u}_\varepsilon)\cdot\boldsymbol{w}\ +\ \varepsilon\partial_t^\tau(hq_\varepsilon)p\right)dy+E\int_0^L\partial_t^\tau u_\varepsilon v\,dy_1\ =\ \int_t^{t+\tau}[\,\dots\,]\,ds \tag{8.5}$$

for a.e. $t\in(0,T-\tau)$. Now we put

$$\boldsymbol{w}=\partial_t^\tau(h\boldsymbol{u}_\varepsilon)\,,\quad p=\partial_t^\tau(hq_\varepsilon)\,,\quad v=\partial_t^\tau\left(\lambda hu_\varepsilon+(1-\lambda)h\frac{\partial h}{\partial t}\right)$$

and integrate (8.5) with respect to t over $(0,T-\tau)$. In this way we arrive at

$$\int_0^{T-\tau}\int_D\left(|\partial_t^\tau(h\boldsymbol{u}_\varepsilon|^2\ +\ \varepsilon\,|\partial_t^\tau(hq_\varepsilon)|^2\right)dy\,dt+\lambda E\int_0^{T-\tau}\int_0^L h\,|\partial_t^\tau u_\varepsilon|^2\,dy_1dt$$

$$= -E\int_0^{T-\tau}\int_0^L \left[\lambda(\partial_t^\tau u_\varepsilon)u_\varepsilon(t+\tau)\partial_t^\tau h + (1-\lambda)(\partial_t^\tau u_\varepsilon)\partial_t^\tau\left(h\frac{\partial h}{\partial t}\right)\right] dy_1 dt$$
$$+ \int_0^{T-\tau}\int_t^{t+\tau}\left[\int_D \ldots\, dy + \int_0^L \ldots\, dy_1 + \int_0^1 \ldots\, dy_2\right] ds\, dt \le c\tau\,.$$

To see that the right hand side does not depend on κ let us focus the corresponding boundary term

$$\kappa\int_0^{T-\tau}\int_0^L [\varphi]_\tau\,(t+\tau)\,\partial_t^\tau(h\varphi)\, dy_1\, dt\,\tau\,,\quad \varphi = u_{2\varepsilon} - \lambda u_\varepsilon - (1-\lambda)\frac{\partial h}{\partial t}\,,$$

where $[\varphi]_\tau\,(y_1, t+\tau) = \frac{1}{\tau}\int_t^{t+\tau}\varphi(y_1,s)\,ds$, see (6.36). As $\|[\varphi]_\tau\|_{L^2(0,T-\tau;L^2(0,L))} \le \|\varphi\|_{L^2(0,T;L^2(0,L))}$ we get

$$\kappa\int_0^{T-\tau}\int_0^L [\varphi]_\tau\,(t+\tau)\,\partial_t^\tau(h\varphi)\, dy_1\, dt\,\tau \le c\kappa\,\|\varphi\|^2_{L^2(0,T;L^2(0,L))}\,\tau$$

that due to (8.1) yields (8.4).

Now, due to the estimates (8.1), (8.4) and the compactness arguments of [2, Lemma 1.9] we may extract a subsequence of $\{\boldsymbol{u}_\varepsilon\}$, $\{u_\varepsilon\}$ which for simplicity we again denote by $\{\boldsymbol{u}_\varepsilon\}$, $\{u_\varepsilon\}$ such that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad\text{in}\quad L^1(D\times(0,T))^2\,,\quad u_\varepsilon \to u \quad\text{in}\quad L^1(\mathcal{S}_T)$$

for $\varepsilon\to 0$. Recall that $\mathcal{S}_T = (0,L)\times(0,T)$. As $\boldsymbol{u}, \{\boldsymbol{u}_\varepsilon\}$ are bounded in $L^4(D\times(0,T))^2$ and $u, \{u_\varepsilon\}$ are bounded in $L^6(\mathcal{S}_T)$, then

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad\text{in}\quad L^p(D\times(0,T))^2\,,\quad u_\varepsilon \to u \quad\text{in}\quad L^q(\mathcal{S}_T) \tag{8.6}$$

strongly, for any $1 \le p < 4$ and $1 \le q < 6$, as $\varepsilon \to 0$.
If $\boldsymbol{\psi}$ is continuously differentiable such that $\boldsymbol{\psi}(T) = 0$ and $\mathrm{div}_h\boldsymbol{\psi} = 0$ we can pass to the limit in (3.21) using the weak convergence results

$$(\boldsymbol{u}_\varepsilon, \varepsilon q_\varepsilon, u_\varepsilon) \longrightarrow (\boldsymbol{u}, 0, u)$$

in $L^2(0,T;V)$ and the strong convergence (8.6). We obtain

$$\int_0^T\int_D\left\{h\boldsymbol{u}\cdot\frac{\partial\boldsymbol{\psi}}{\partial t} + \frac{\partial h}{\partial t}\frac{\partial(y_2\boldsymbol{u})}{\partial y_2}\cdot\boldsymbol{\psi}\right\} dy\, dt \tag{8.7}$$
$$= \int_0^T \Bigg\{ \nu\,((\boldsymbol{u},\boldsymbol{\psi}))_h + b_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi})$$
$$+ R\left(\int_0^1 q_{out}(y_2,t)\psi_1\,(L,y_2,t)\,dy_2 - \int_0^1 q_{in}(y_2,t)\psi_1\,(0,y_2,t)\,dy_2\right)$$
$$+ \int_0^L\left(q_w + \frac{1}{2}\frac{\partial h}{\partial t}u_2\right)\psi_2\,(y_1,1,t)\,dy_1$$

$$+\kappa\int_0^L\left(u_2-\lambda u-(1-\lambda)\frac{\partial h}{\partial t}\right)\left(\psi_2\,(y_1,1,t)-\frac{\xi}{E}(y_1,t)\right)\,dy_1$$
$$+\int_0^L\left(-u\frac{\partial\xi}{\partial t}+c\,\frac{\partial u}{\partial y_1}\frac{\partial\xi}{\partial y_1}+a\frac{\partial}{\partial y_1}\int_0^t u(y_1,s)ds\frac{\partial\xi}{\partial y_1}\right.$$
$$\left.\left.+b\int_0^t u(y_1,s)ds\xi\right)(y_1,t)\,dy_1\right\}dt\,,$$

where $b_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi})$ is defined by the right hand side of (4.3) with $h,\boldsymbol{u}$ instead of $h^i,\boldsymbol{u}^i$.
Due to the approximation arguments we can conclude that (8.7) holds for every $(\boldsymbol{\psi},\xi)\in L^2(0,T;\boldsymbol{V}\times H_0^1(0,L))),\boldsymbol{\psi}\in L^4(0,T;L^4(D)^2)$ and $\mathrm{div}_h\boldsymbol{\psi}=0$.

We continue this section by making precise the meaning of a solution of our original Problem (1.1)–(1.10).

Definition 8.1 *A couple $(\boldsymbol{v},\eta)\in L^2(\Omega(h))^2\times L^2(\mathcal{S}_T)$ is called a weak solution of the initial boundary value problem (1.1)–(1.10) if $(\boldsymbol{u},u)$ defined by (3.3) and (3.5) fulfill the following two conditions:*
1) $(\boldsymbol{u},u)\in L^\infty(0,T;L^2(D)^2\times L^2(0,L))\cap L^2(0,T;\boldsymbol{V}\times H_0^1(0,L))$,
2) $\boldsymbol{u}$ satisfies the condition (8.2) and the integral identity (8.7) holds.

Consequently, we have proved the following statement.

Theorem 8.1 *Let*

$$h\in W^{1,\infty}(\mathcal{S}_T)\cap W^{2,2}(\mathcal{S}_T)\quad\text{satisfy (1.2)}\quad\text{and}$$
$$p_{in}\,,\ p_{out}\in L^2(0,T;L^2(0,1))\,,\ p_w\in L^2(0,T;L^2(0,L))\,.$$

Then there exist a weak solution $(\boldsymbol{v},\ \eta)$ of Problem (1.1)–(1.10) in the sense of Definition 8.1.

Proof. Take $(\boldsymbol{u},u)$ as a limit of $(\boldsymbol{u}_\varepsilon,u_\varepsilon)$ and define

$$\boldsymbol{v}(x_1,x_2,t)\overset{\text{def}}{=}\boldsymbol{u}\left(x_1,\frac{x_2}{h(x_1,t)},t\right)\qquad\text{and}\qquad\eta(x_1,t)\overset{\text{def}}{=}\int_0^t u(x_1,s)\,ds\,.$$

Then $(\boldsymbol{v},\eta)$ is a desired solution of Problem (1.1)–(1.10). ■

Remark 8.1. It is easy to check that (8.7) can be rewritten into

$$\int_{\Omega(h)}\left\{-\boldsymbol{v}\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}+\nu\nabla\boldsymbol{v}\cdot\nabla\boldsymbol{\varphi}+\sum_{i,j=1}^2 v_i\frac{\partial v_j}{\partial x_i}\varphi_j\right\}dxdt$$
$$+\int_0^T\int_0^R\left(p_{out}-\frac{1}{2}\,|v_1|^2\right)\varphi_1(L,x_2,t)\,dx_2dt$$
$$-\int_0^T\int_0^R\left(p_{in}-\frac{1}{2}\,|v_1|^2\right)\varphi_1(0,x_2,t)\,dx_2dt$$

$$+\int_0^T\int_0^L\left(p_w-\frac{1}{2}u_2\left(v_2-\frac{\partial h}{\partial t}\right)\right)\varphi_2(x_1,h(x_1,t),t)\,dx_1dt$$
$$+\kappa\int_0^T\int_0^L\left(v_2-\lambda u-(1-\lambda)\frac{\partial h}{\partial t}\right)\left(\varphi_2(x_1,h(x_1,t),t)-\frac{\xi}{E}(x_1,t)\right)dx_1dt$$
$$+\int_0^T\int_0^L\left(-\frac{\partial\eta}{\partial t}\frac{\partial\xi}{\partial t}+c\,\frac{\partial^2\eta}{\partial x_1\partial t}\frac{\partial\xi}{\partial x_1}+a\frac{\partial\eta}{\partial x_1}\frac{\partial\xi}{\partial x_1}+b\eta\,\xi\right)(x_1,t)\,dx_1dt\;=\;0\,.$$

□

According to a priori estimates (8.1) and (8.4) that are independent on κ we can let $\kappa^{-1}=\varepsilon\to 0$ in (3.21) for a special choice of a test function ξ satisfying

$$\xi(y_1,t)\equiv E\psi_2(y_1,1,t)\,.$$

In this case terms with κ cancel and we let $\varepsilon\to 0$ to find

$$\int_0^T\int_D\left\{h\boldsymbol{u}\cdot\frac{\partial\boldsymbol{\psi}}{\partial t}+\frac{\partial h}{\partial t}\frac{\partial(y_2\boldsymbol{u})}{\partial y_2}\cdot\boldsymbol{\psi}\right\}dydt \tag{8.8}$$
$$=\int_0^T\Bigg\{\nu\,(\!(\boldsymbol{u},\boldsymbol{\psi})\!)_h+b_h(\boldsymbol{u},\boldsymbol{u},\boldsymbol{\psi})$$
$$+R\left(\int_0^1 q_{out}(y_2,t)\psi_1(L,y_2,t)dy_2-\int_0^1 q_{in}(y_2,t)\psi_1(0,y_2,t)dy_2\right)$$
$$+\int_0^L\left(q_w+\frac{1}{2}\frac{\partial h}{\partial t}u_2\right)\psi_2(y_1,1,t)\,dy_1$$
$$+\int_0^L\Bigg(-u\frac{\partial\xi}{\partial t}+c\,\frac{\partial u}{\partial y_1}\frac{\partial\xi}{\partial y_1}+a\frac{\partial}{\partial y_1}\int_0^t u(y_1,s)ds\frac{\partial\xi}{\partial y_1}$$
$$+b\int_0^t u(y_1,s)ds\,\xi\Bigg)(y_1,t)\,dy_1\Bigg\}dt\,.$$

By the same procedure as before one can prove the following

Theorem 8.2 *Let* $\lambda=1$, $h\in W^{1,\infty}(\mathcal{S}_T)$ *satisfy (1.2) and* p_{in} , $p_{out}\in L^2(0,T;L^2(0,1))$, $p_w\in L^2(0,T;L^2(0,L))$.

Then there exists a couple $(\boldsymbol{v},\eta)\in L^2(\Omega(h))^2\times L^2(\mathcal{S}_T)$ *such that* $(\boldsymbol{u},u)$ *defined by (3.3) and (3.5) fulfill the following two conditions:*
1) $(\boldsymbol{u},u)\in L^\infty(0,T;L^2(D)^2\times L^2(0,L))\cap L^2(0,T;\boldsymbol{V}\times H_0^1(0,L))$,
2) $\boldsymbol{u}$ *satisfies the condition (8.2) and the integral identity (8.8) holds.*

In view of Theorem 8.2, $(\boldsymbol{v},\eta)$ is a weak solution of the Problem (1.1)–(1.10), in which (1.4) and (1.5) are replaced by

$$\left[\mu\frac{\partial v_2}{\partial x_1}\left(-\frac{\partial h}{\partial x_1}\right)+\mu\frac{\partial v_2}{\partial x_2}-p+p_W-\frac{\rho}{2}v_2\left(v_2-\frac{\partial h}{\partial t}\right)\right](x_1,h(x_1,t),t) \tag{8.9}$$
$$=-E\left[\frac{\partial^2\eta}{\partial t^2}-a\frac{\partial^2\eta}{\partial x_1^2}+b\eta-c\frac{\partial^3\eta}{\partial t\partial x_1^2}\right](x_1,t)$$

and

$$v_2(x_1, h(x_1,t), t) = \frac{\partial \eta}{\partial t}(x_1, t) \tag{8.10}$$

for almost all $0 < x_1 < L$, $0 < t < T$.

9. Numerical experiments

Finally, let us present some numerical experiments for our problem with the periodic inflow pressure $p_{in} = 6 + 10\sin(2\pi t)\, dynes.cm^{-2}$, outflow pressure $p_{out} = 0$, external pressure acting on the elastic wall $p_w = 0$

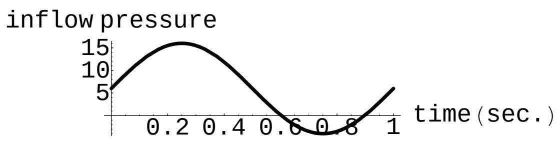

Figure 1: Inflow pressure function

and the following experimental data, see e.g. [12]:

Kinematic viscosity $\nu = \rho^{-1}\mu = 0.09$ poise, density of the fluid $\rho = 1\, g.cm^{-3}$
Reference radius $R = 0.5\, cm$
Tube length $L = 5.1\, cm$
Time interval $(0, T)$, $T = 4\, s$
Time step $\Delta t = 0.005\, s$
Density of the vessel wall tissue $\rho_w = 1.1\, g.cm^{-3}$
Young's modulus $\mathcal{E} = 0.75.10^3\, dynes.cm^{-2}$
Timoshenko's shear correction factor $\Theta = 1$
Shear modulus $G = \mathcal{E}/2(1+\delta)$, $\delta = 0.5$ for incompressible materials
Wall thickness $h = 0.09\, cm$
Viscoelasticity coefficient $\gamma = 4.0$

The physical meaning of the constants a, b, c, E from the condition (1.5) is as follows:

$$a = \frac{\Theta G}{\rho_w}, \qquad b = \frac{\mathcal{E}}{\rho_w R^2}, \qquad c = \frac{\gamma}{\rho_w h}, \qquad E = \rho_w h\,.$$

The UG software toolbox [3] (UG problem class for mowing domain [5]) is used to solve the velocity field in a time dependent domain. The finite volume method is implemented in UG, see e.g. [17]. For solving the interface condition (1.5) the implicit finite difference scheme is used. We have performed two types of experiments.

1. We have executed experiments for the problem (2.10)–(2.12), where some iteration in k have been done, i.e. $h(x,t) = R + \eta^{(k)}(x,t)$, $\eta^0 = 0$. Here κ is kept fixed. The numerical solution for $k = 1$, i.e., $h(x,t) = R + \eta^{(1)}(x,t)$ one can see

on the Figure 2 for $\kappa = 100$.

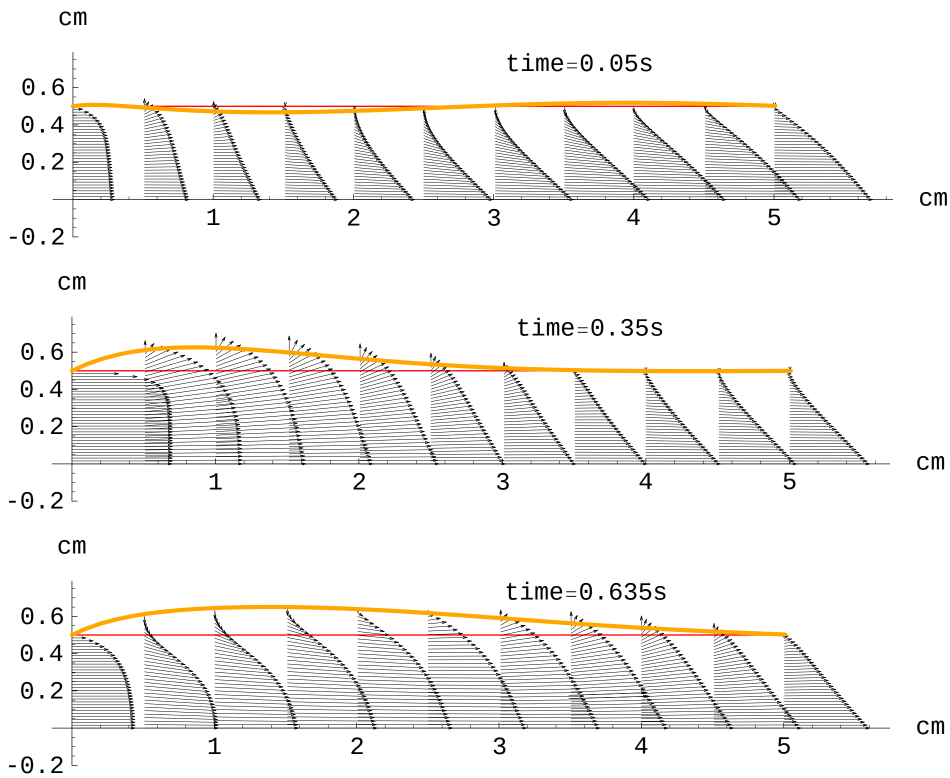

Figure 2: Velocity field in a compliant domain for $\kappa = 100$

A point-wise convergence of $\eta^{(k)}$ (the domain deformation) can be observed after a few iterations, see Figure 3 and Table 1.

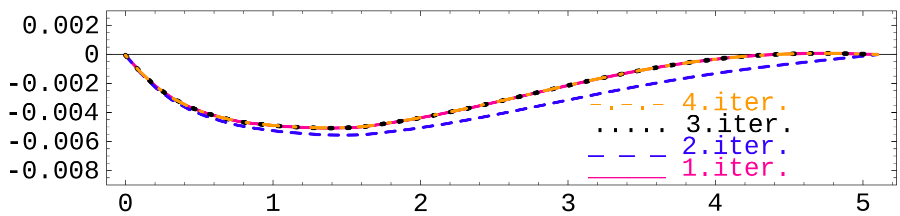

Figure 3: Convergence of the wall deformation for $\kappa = 12$, time=2.95 s

Values of the deformation at time t=2.95 s		
Iteration	Deformation values (in cm) for $\kappa = 12$	
	x=1.02 cm	x=3.634 cm
1.	-0.00481124167725	-0.00060430218306
2.	-0.00528328152442	-0.00179846044021
3.	-0.00491893241581	-0.00071200844506
4.	-0.00491116991577	-0.00071094334787
11.	-0.00491118094564	-0.00071051744755

Table 1: Convergence of the function $\eta^{(k)}$.

2. We have also performed experiments for problem (2.10)–(2.12) for various κ. An interesting observation can be made by an increasing κ. Note that if $\kappa \to \infty$, then our problem of fluid flow in time-dependent domain with a given domain deformation $h(x,t)$ particularly corresponds to the problem proposed by A. Quarteroni in [20], see Section 2. One can see (Figure 4,5 and 6), that for increasing κ the domain deformation increases. Moreover, the fluid flows into the tube and out of the tube through the upper, 'permeable' and deforming boundary Γ_w slower.

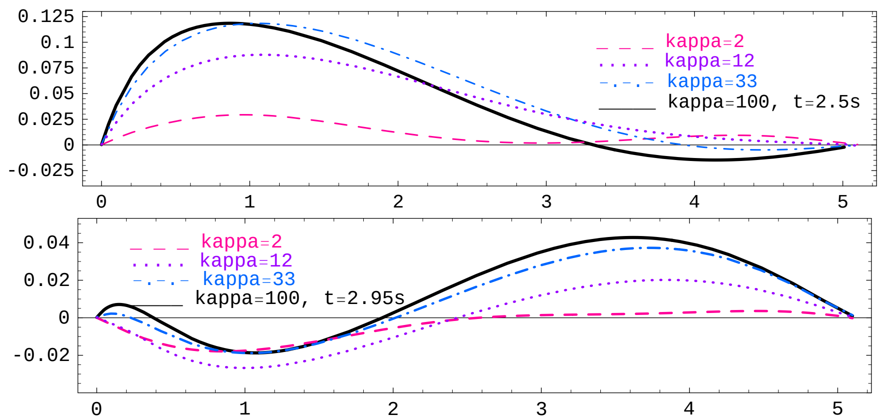

Figure 4: Comparison of the moving wall deformations for different κ

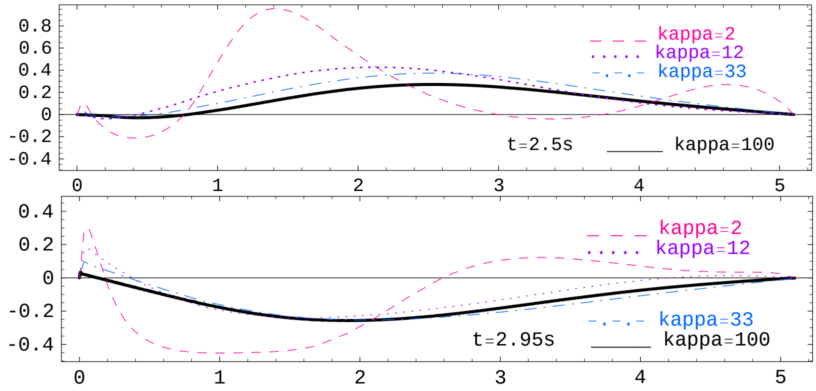

Figure 5: Comparison of fluid velocities on the deformed boundary for different κ

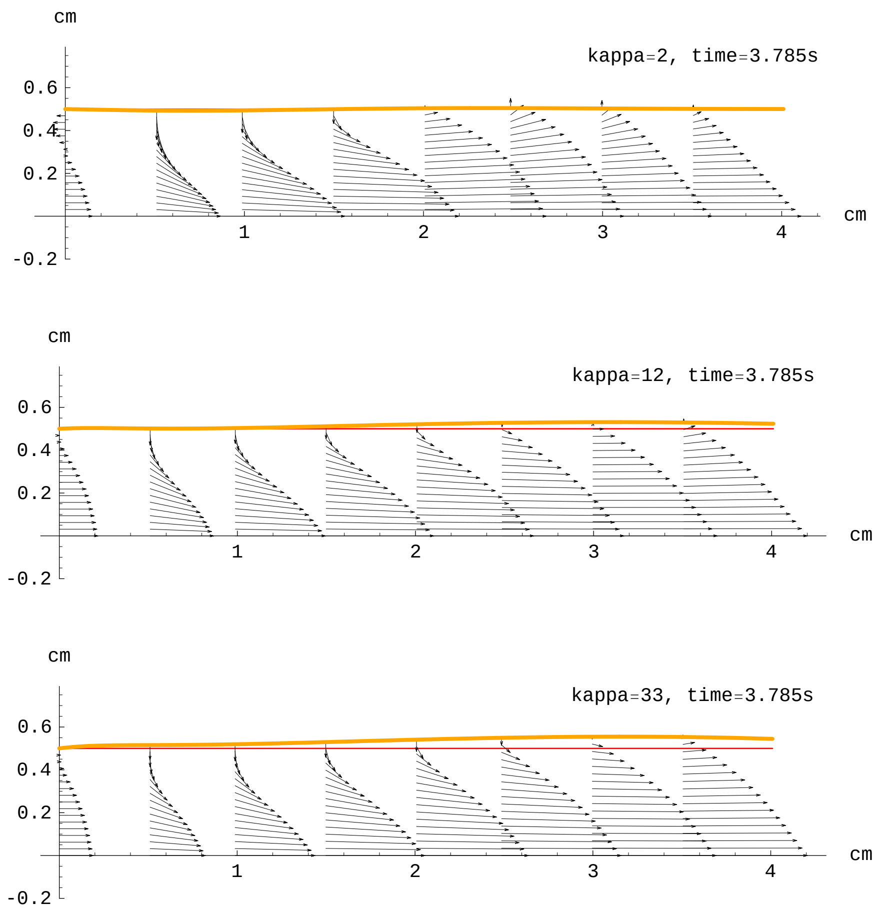

Figure 6: Velocity field for varying κ in time=3.875 s

10. Global iteration

Let us finish our paper with some comments to our approximation steps described in Section 2. In Theorem 6.3 and Theorem 7.1 we prove the existence and uniqueness result for the problem with given h and fixed positive ε, κ. Theorem 8.1 provides the existence result for given h and κ, letting $\varepsilon \to 0$. Finally, Theorem 8.2 presents the existence of a solution for fixed h sending $\kappa = \varepsilon^{-1} \to \infty$. In the original problem, however, h is not known and is to be found.

In this final section we show that $\{(\boldsymbol{v}^{(k)}, p^{(k)}, \eta^{(k)})\}_{k=1}^{\infty}$ converges, but only for fixed positive ε, κ and for $a = b = 0$, $\lambda = 1$. We begin by supposing

$$\eta^{(k)} \in \mathcal{X} \equiv \left\{ \eta \in \mathcal{M} :\ \|\eta\|_{W^{1,\infty}(\mathcal{S}_T)} \le R - \alpha \right\}, \quad \mathcal{S}_T = (0,L) \times (0,T), \qquad (10.1)$$

for a positive constant $\alpha < \min\{1, R\}$ such that $(2R - \alpha)\alpha \le 1$, in which

$$\mathcal{M} = \left\{ \eta \in W^{1,\infty}(\mathcal{S}_T) :\ \eta(\cdot,t) \in W_0^{1,\infty}(0,L) \quad \text{and} \quad \eta(\cdot,0) = 0 \right\}.$$

According to Theorem 6.3, for given $h^{(k)} = R + \eta^{(k)}$, q_{in}, q_w, q_{out} there exists a unique weak solution $(\boldsymbol{u}^{(k+1)}, q^{(k+1)}, \eta_t^{(k+1)})$ of Problem (3.6)–(3.16). Moreover, the a priori estimates (8.1) and (8.4) hold. Observe that these estimates depend on α, but are independent on κ, ε.

Next, let $(\boldsymbol{u}^{(k+1)}, q^{(k+1)}, \eta_t^{(k+1)})$ and $(\tilde{\boldsymbol{u}}^{(k+1)}, \tilde{q}^{(k+1)}, \tilde{\eta}_t^{(k+1)})$ be weak solutions of Problems (3.6)–(3.16) with given functions $h^{(k)} = R + \eta^{(k)}$, q_{in}, q_w, q_{out} and $\tilde{h}^{(k)} = R + \tilde{\eta}^{(k)}$, q_{in}, q_w, q_{out}, respectively, and suppose that $\eta^{(k)}, \tilde{\eta}^{(k)} \in \mathcal{X}$. Theorem 7.1 then yields

$$\begin{aligned} &\int_D \left| h^{(k)} \boldsymbol{u}^{(k+1)} - \tilde{h}^{(k)} \tilde{\boldsymbol{u}}^{(k+1)} \right|^2 (t) + \nu \int_0^T \int_D \left| \nabla \left(h^{(k)} \boldsymbol{u}^{(k+1)} - \tilde{h}^{(k)} \tilde{\boldsymbol{u}}^{(k+1)} \right) \right|^2 \\ &\le \omega(T) \left\| \eta^{(k)} - \tilde{\eta}^{(k)} \right\|^2_{W^{1,\infty}(\mathcal{S}_T)} \end{aligned} \qquad (10.2)$$

for any $t \in [0,T]$, where $\omega(T) \downarrow 0$ as $T \to 0$. Observe that the function ω depends also on α, κ.

One way how to prove the convergence of this procedure with respect to (k) should be an application of Banach's fixed point theorem. We will try to apply Banach's theorem in the space $\mathcal{X}$ with the norm of $W^{1,\infty}(S_T)$ if $T > 0$ is small enough. Let the operator A be defined as follows. Given a function $\eta^{(k)} \in \mathcal{X}$ define $A : \mathcal{X} \to \mathcal{X}$ by setting $A\left[\eta^{(k)}\right] = \eta^{(k+1)}$.

We now claim that if $T > 0$ and the L^2-norm of q_S (see (7.2)) are small enough, then A is well defined and it is a strict contraction. To prove this, we apply the classical results concerning the regularity of the solutions of the following linear parabolic equations

$$\eta_{tt} - c\,\eta_{tyy} + d\,\eta_t = d\, u_2(y,1,t) \qquad \text{on } \mathcal{S}_T \qquad (10.3)$$

$$\eta_t - c\,\eta_{yy} + d\,\eta = d \int_0^t u_2(y,1,s)ds \qquad \text{on } \mathcal{S}_T \qquad (10.4)$$

$$\eta(0,t) = \eta_t(0,t) = \eta(L,t) = \eta_t(L,t) = 0 \qquad \eta(y,0) = \eta_t(y,0) = 0$$

from [15], where $d = E^{-1}\kappa$ Referring to Proposition 3.2 and [15, Theorem 2.1 of Chapter II], we have the following regularity of the right hand sides of (10.3) and (10.4):

$$u_2 \in L^3(S_w \times (0,T)) \quad \text{and} \quad \int_0^t u_2 \in L^4(S_w \times (0,T)) \, .$$

Therefore, taking into account [15, Theorem 9.1 of Chapter IV], we see that

$$\eta_{tt} \, , \; \eta_{ty} \, , \; \eta_{tyy} \, \in \, L^3(\mathcal{S}_T) \quad \text{and} \quad \eta_t \, , \; \eta_y \, , \; \eta_{yy} \, \in \, L^4(\mathcal{S}_T) \, ,$$

and that there exists a positive constant C such that

$$\|\eta\|_2 + \|\eta_t\|_2 + \|\eta_y\|_2 + \|\eta_{tt}\|_2 + \|\eta_{ty}\|_2 + \|\eta_{yy}\|_2 + \|\eta_{tyy}\|_2 \, \leq \, C \, \|f\|_2 \qquad (10.5)$$

where $\| \cdot \|_2 = \| \cdot \|_{L^2(\mathcal{S}_T)}$ and $f(y,t) = d \, u_2(y,1,t)$. Now, according to the standard manipulation, (10.3),(10.4) and (10.5) yield the existence of a positive constant C_1 such that

$$\|\eta\|_{W^{1,\infty}(S_T)} \, \leq \, C_1 \, \|f\|_2 \, . \qquad (10.6)$$

Indeed, we first multiply (10.3) by η_{yy} and integrate over S_t to discover

$$\sup_{0 \leq t \leq T} \frac{c}{2} \int_0^L |\eta_{yy}(y,t)|^2 \, dy \, \leq \, \|\eta_{yy}\|_2 \left(\|\eta_{tt}\|_2 \, + \, \|f\|_2 \right) \, .$$

Similarly, we multiply (10.4) by η_t and we find

$$\sup_{0 \leq t \leq T} \frac{c}{2} \int_0^L |\eta_y(y,t)|^2 \, dy \, \leq \, \|f\|_2 \|\eta\|_2 \, + \, \|\eta_t\|_2^2 \, .$$

Finally, we multiply (10.3) by η_{tt} to deduce

$$\sup_{0 \leq t \leq T} \frac{c}{2} \int_0^L |\eta_{ty}(y,t)|^2 \, dy \, \leq \, \|\eta_{tt}\|_2 (\|f\|_2 \, + \, \|\eta_{tt}\|_2)$$

and the stated estimate (10.6) follows easily due to the embedding $W^{1,2}(0,L) \subset C^{0,1/2}([0,L])$ with the assistance of (10.5). In view of the estimates (8.1) and (10.6) we have the estimate

$$\left\| A \left[\eta^{(k)} \right] \right\|^2_{W^{1,\infty}(\mathcal{S}_T)} \, \leq \, C_2 \alpha^{-1} (2 + (R - \alpha)^2) \, \|q_S\|^2_{L^2(S_T)} \, . \qquad (10.7)$$

Similar arguments show

$$\|\eta^{(k+1)} - \tilde{\eta}^{(k+1)}\|_{W^{1,\infty}(S_T)} \, \leq \, C_3 \|u_2^{(k+1)} - \tilde{u}_2^{(k+1)}\|_2 \, . \qquad (10.8)$$

So in order that (10.2) can help us to estimate the right hand side of (10.8), let us observe

$$\|u_2^{(k+1)} - \tilde{u}_2^{(k+1)}\|_2 \leq \alpha^{-1} \|h^{(k)} u_2^{(k+1)} - \tilde{h}^{(k)} \tilde{u}_2^{(k+1)}\|_2 + \|h^{(k)} - \tilde{h}^{(k)}\|_{L^\infty(\mathcal{S}_T)} \|\tilde{u}_2^{(k+1)}\|_2$$

and we arrive at

$$\left\| A\left[\eta^{(k)}\right] - A\left[\tilde{\eta}^{(k)}\right]\right\|_{W^{1,\infty}(\mathcal{S}_T)} \leq \varpi(T)\left\|\eta^{(k)} - \tilde{\eta}^{(k)}\right\|_{W^{1,\infty}(\mathcal{S}_T)}$$

for $\varpi(T) \to 0$ as $T \to 0$ (see 7.10). Thus, A maps $\mathcal{X}$ into $\mathcal{X}$ and A is a contraction provided $T > 0$ and $\|q_S\|_{L^2(S_T)}$ are so small that

$$\sqrt{C_2\alpha^{-1}(2 + (R-\alpha)^2)}\;\|q_S\|_{L^2(S_T)} \leq R - \alpha \quad \text{and} \quad \varpi(T) < 1\,. \tag{10.9}$$

Then A has one and only one fixed point $\eta^* \in \mathcal{X}$ and for any $\eta^{(0)} \in \mathcal{X}$ the sequence $\eta^{(k+1)} = A[\eta^{(k)}]$ coverges to the fixed point η^* in $W^{1,\infty}(\mathcal{S}_T)$. Consequently, for $h^* = R + \eta^*$ and fixed κ, say $\kappa = \varepsilon^{-1}$, $0 < \varepsilon \ll 1$ there is a unique solution $(\boldsymbol{u}^*, q^*, \eta_t^*) = (\boldsymbol{u}_\varepsilon^*, q_\varepsilon^*, (\eta_\varepsilon^*)_t)$ of Problem (3.6)-(3.16) in the sense of Definition 3.1 such that (6.33) holds.

Let us now summarize the main result of this section.

Theorem 10.1 *Let us consider Problem (3.6)-(3.16) for unknown*

$$(\boldsymbol{u}, q, u) \quad \text{and} \quad h = R + \eta$$

with given positive constants $R, L, T, \nu, \varepsilon, c, \kappa, E$ and let $a = b = 0, \lambda = 1$. Moreover, let positive α satisfies (10.1) and q_{in}, q_{out}, q_w and T are so small that (10.9) holds. Then there exists a unique solution $(\boldsymbol{u}, q, u)$ of Problem (3.6)-(3.16) in the sense of Definition 3.1 *and $\eta \in \mathcal{X}$ such that*

$$h = R + \eta \quad \text{and} \quad u = \frac{\partial \eta}{\partial t}\,.$$

Remark 10.1. *Note that the obtained solution depends on ε, κ. In order to be able to let $\varepsilon \to 0$ and $\kappa \to \infty$ and to arive at the original problem we need, besides others, at least $W^{1,\infty}$ convergence of $\eta = \eta_{\varepsilon,\kappa}$. This remains open.*

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