HIERARCHICAL SUBSTRUCTURING COMBINED WITH SVD-BASED MODEL REDUCTION METHODS

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Abstract. The direct applicability of SVD-based methods in model reduction of large linear systems is very limited. However, substructuring methods are a possibility to use these approaches. A method called Automated Multilevel Substructuring (AMLS) has been successfully applied to eigenvalue computations of very large systems. We present a similar substructuring approach for linear time-invariant (LTI) systems and its combination with model reduction techniques. Because the reduction methods are only applied to arising significantly smaller subsystems, particularly SVD-based methods are used.

1. Introduction. Model reduction techniques play an important role in simulation, control and optimization of physical processes. Often the modelling of such processes yields large-scale systems, e.g., due to discretizations of partial differential equations. Hence, order reduction becomes necessary for computational feasibility.
SVD-based methods are a very popular class of reduction techniques. Due to their computational effort it is difficult to apply them to large systems.
We combine SVD-based methods with substructuring techniques that decompose the given problem into several subproblems. Using hierarchical substructuring we obtain smaller interface substructures than by using single-level methods like CMS ([6]). In contrast to CMS order reduction is also applied to interface substructures.
In section 2 we use the theory of parabolic differential equations to explain what is done by using the method on the discrete level. After a short review of reduction methods in section 3 we state our reduction framework in section 4. Section 5 provides results regarding the error of the method. Finally we apply it to two benchmark examples.
For our purposes Ω is a domain of $\mathbb{R}^d$ ($d \in \{1, 2\}$) with Lipschitz boundary $\partial\Omega$ and $f \in L^2(\Omega \times (0,T))$. The standard Sobolev space is denoted by $H^1(\Omega)$ and $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. The appropriate trace space is denoted by $H^{1/2}(\partial\Omega)$. Restrictions of functions to boundaries are meant in the sense of trace operators. Furthermore we denote by V' the dual of a vector space V and by $\langle \cdot, \cdot \rangle$ the duality pairing.

2. Substructuring. In this section we consider substructuring techniques applied to parabolic partial differential equations. Most of these approaches are described in [16]. The application of substructuring techniques in the analysis of AMLS for eigenvalue problems is developed in [5], where elliptic problems are discussed. We adapt the ideas given there for parabolic problems. The first part considers only a single level of substructuring with two subdomains to illustrate the main ideas. The second part extends the results to the multi-level case.

2.1. Single-level case. Consider the model problem

$$\begin{aligned} \frac{\partial}{\partial t} u + Lu &= f, \text{ in } \Omega \times (0,T) \\ u &= 0, \text{ in } \partial\Omega \times (0,T) \\ u|_{t=0} &= 0, \text{ in } \Omega, \end{aligned} \tag{2.1}$$

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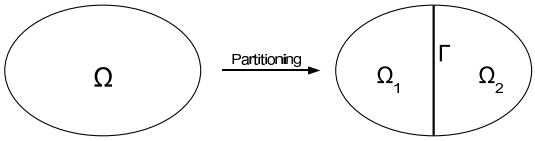


FIG. 2.1. *Decomposition of* Ω

where L is a symmetric elliptic differential operator with respect to x.

A standard variational formulation of (2.1) yields:
Find $u \in L^2((0,T), H_0^1(\Omega))$, *s.t.*

$$\frac{d}{dt}(u(t),v)_\Omega + a_\Omega(u(t),v) = (f(t),v)_\Omega,\ \forall v \in H_0^1(\Omega) \tag{2.2}$$

$$u(0) = 0, \tag{2.3}$$

where $\frac{d}{dt}u \in L^2\left((0,T), H_0^1(\Omega)'\right)$ denotes the weak derivative of u, $a_\Omega(\cdot,\cdot)$ is a coercive and continuous bilinear form, and $(u,v)_\Omega := \int_\Omega uvdx$. See [14] and [16] for more details.

We divide the domain Ω into two disjoint subdomains Ω_i, $i = 1,2$, with the interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ (figure 2.1) and define the space $\Lambda := \{\lambda \in H^{1/2}(\Gamma) \mid \lambda = v|_\Gamma, v \in H_0^1(\Omega)\}$. Our aim is to formulate adequate parabolic problems on the subdomains and the interface. Therefore let $\mathcal{E}_i\lambda \in V_i := \{v \in H^1(\Omega_i) \mid v|_{\partial\Omega\cap\partial\Omega_i} = 0\}$ be the *harmonic extension* of $\lambda \in \Lambda$ (see [16]), i.e.,

$$\begin{aligned} a_{\Omega_i}(\mathcal{E}_i\lambda, v) &= 0,\ \forall v \in H_0^1(\Omega_i) \\ \mathcal{E}_i\lambda|_\Gamma &= \lambda. \end{aligned}$$

By $\mathcal{E}\lambda := \begin{cases} \mathcal{E}_1\lambda, & \text{on } \Omega_1 \\ \mathcal{E}_2\lambda, & \text{on } \Omega_2 \end{cases}$ we denote the extension onto the entire domain. We define

$$V_i^0 := \{v \in H_0^1(\Omega) \mid v|_{\Omega\setminus\Omega_i} = 0\}, i = 1,2 \tag{2.4}$$

$$V_\Gamma := \{v \in H_0^1(\Omega) \mid v = \mathcal{E}\lambda, \lambda \in \Lambda\}. \tag{2.5}$$

An immediate consequence of these definitions proven in [5] is a decomposition of the space $H_0^1(\Omega)$.

COROLLARY 2.1 ([5]).

$$H_0^1(\Omega) = V_1^0 \oplus V_\Gamma \oplus V_2^0 \tag{2.6}$$

is an orthogonal decomposition of $H_0^1(\Omega)$ *w.r.t. the bilinear form* $a_\Omega(\cdot,\cdot)$.

Hence the solution to (2.2)-(2.3) can be splitted into three parts with (2.4) and (2.5) as range spaces. Therefore it is evident to formulate subproblems with respect to this orthogonal decomposition. The two parabolic problems on the subdomains are given by:
Find $u_i \in L^2((0,T); H_0^1(\Omega_i))$, $i = 1,2$, *s.t.*

$$\begin{aligned} \frac{d}{dt}(u_i(t),v)_{\Omega_i} + a_{\Omega_i}(u_i(t),v) &= (f(t),v)_{\Omega_i},\ \forall v \in H_0^1(\Omega_i) \\ u(0) &= 0, \end{aligned} \tag{2.7}$$

and the interface problem is defined by:
Find $\lambda \in L^2((0,T);\Lambda)$, *s.t.*

$$\begin{aligned} \frac{d}{dt}(\mathcal{E}\lambda(t),\mathcal{E}\mu)_\Omega + a_\Omega(\mathcal{E}\lambda(t),\mathcal{E}\mu) &= (f(t),\mathcal{E}\mu)_\Omega,\ \forall \mu \in \Lambda \\ \lambda(0) &= 0. \end{aligned} \tag{2.8}$$

It can be shown that problem (2.8) has a unique solution. Note that this system of substructures is not equivalent to (2.2)-(2.3). However, it is a reasonable choice, if one wants to define three independent subproblems induced by the subdomains and the interface.

The interface problem (2.8) can be reformulated by use of linear operators. Let $S : \Lambda \to \Lambda'$ be the Steklov-Poincaré operator defined by

$$\langle S\nu,\mu\rangle := \sum_{i=1}^{2} a_{\Omega_i}(\mathcal{E}_i\nu,\mathcal{E}_i\mu),\ \forall \nu,\mu \in \Lambda. \tag{2.9}$$

In addition we define the operator $\chi : L^2(\Omega) \to \Lambda'$ by

$$\langle \chi(\zeta),\mu\rangle := (\zeta,\mathcal{E}\mu)_\Omega,\ \mu \in \Lambda. \tag{2.10}$$

With these preliminaries we can state the alternative formulation of problem (2.8).

THEOREM 2.2. *The parabolic problem* (2.8) *is equivalent to:*
Find $\lambda \in L^2((0,T);\Lambda)$, *s.t.*

$$\begin{aligned} \frac{d}{dt}\langle \mathcal{M}\lambda(t),\mu\rangle + \langle \mathcal{S}\lambda(t),\mu\rangle &= \langle \chi(f),\mu\rangle,\ \forall \mu \in \Lambda \\ \lambda(0) &= 0, \end{aligned} \tag{2.11}$$

where for $\mu \in \Lambda$ *the mass operator* $\mathcal{M}$ *is given by*

$$\mathcal{M}\mu := \chi(\mathcal{E}\mu).$$

Proof. For almost every $t \in (0,T)$ it holds $\lambda(t) \in \Lambda$. Hence for $\mu \in \Lambda$

$$\begin{aligned} (\mathcal{E}\lambda(t),\mathcal{E}\mu)_\Omega &= \langle \chi(\mathcal{E}\lambda(t)),\mu\rangle \\ &= \langle \mathcal{M}\lambda(t),\mu\rangle,\ \text{for } t \in (0,T) \text{ a.e.} \end{aligned}$$

Thus the two mappings

$$\begin{aligned} &t \mapsto (\mathcal{E}\lambda(t),\mathcal{E}(\cdot))_\Omega \text{ and} \\ &t \mapsto \langle \mathcal{M}\lambda(t),\cdot\rangle \end{aligned}$$

are equivalent in $L^2((0,T);\Lambda')$ and therefore they have the same distributive derivative. The equality of the remaining terms follows by the definitions of S and χ and analog arguments. □

Remark: The mass operator $\mathcal{M}$ was already introduced in [5]. For more detailed information about the operators S and χ and their strong forms see, e.g., [16].

By an FE-semi-discretization w.r.t x of (2.2) and (2.3) we obtain the initial value problem

$$\underbrace{\begin{pmatrix} M_1 & & M_{1,\Gamma} \\ & M_2 & M_{2,\Gamma} \\ M_{\Gamma,1} & M_{\Gamma,2} & M_\Gamma \end{pmatrix}}_{=:M} \frac{du}{dt} + \underbrace{\begin{pmatrix} A_1 & & A_{1,\Gamma} \\ & A_2 & A_{2,\Gamma} \\ A_{\Gamma,1} & A_{\Gamma,2} & A_\Gamma \end{pmatrix}}_{=:A} u = \begin{pmatrix} f_1 \\ f_2 \\ f_\Gamma \end{pmatrix},\ u(0) = 0. \tag{2.12}$$

Note that the arrow-head structure of the matrices is a consequence of the partitioning of Ω, because the subdomains are only coupled with the interface. We eliminate the off-diagonal blocks of the stiffness matrix by block elimination to obtain the equivalent system

$$\begin{pmatrix} M_1 & & \tilde{M}_{1,\Gamma} \\ & M_2 & \tilde{M}_{2,\Gamma} \\ \tilde{M}_{\Gamma,1} & \tilde{M}_{\Gamma,2} & M_d \end{pmatrix} \frac{d\tilde{u}}{dt} + \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & S_d \end{pmatrix} \tilde{u} = \begin{pmatrix} f_1 \\ f_2 \\ \chi_d(f) \end{pmatrix}, \ \tilde{u}(0) = 0, \tag{2.13}$$

where

$$S_d := A_\Gamma - A_{\Gamma,1}A_1^{-1}A_{1,\Gamma} - A_{\Gamma,2}A_2^{-1}A_{2,\Gamma}, \tag{2.14}$$

$$M_d := M_\Gamma - \sum_{i=1}^{2} \left(A_{\Gamma,i}A_i^{-1}M_{i,\Gamma} + M_{\Gamma,i}A_i^{-1}A_{i,\Gamma} - A_{\Gamma,i}A_i^{-1}M_iA_i^{-1}A_{i,\Gamma} \right) \tag{2.15}$$

and

$$\chi_d(f) := f_\Gamma - A_{\Gamma,1}A_1^{-1}f_1 - A_{\Gamma,2}A_2^{-1}f_2. \tag{2.16}$$

The block entries S_d, M_d and $\chi_d(f)$ are discretizations of $\mathcal{S}, \mathcal{M}$ and $\chi(f)$. Hence, if we ignore the coupling blocks in the mass matrix of (2.13), we obtain three subsystems which are discretizations of the subdomain problems (2.7) and the interface problem (2.11).

2.2. Multi-level case. The principles used in the single-level version can directly be extended to the multi-level case. The j-th subdomain on the i-th level is denoted by $\Omega_{i,j}$ with $\Omega_{0,1} := \Omega$. The domains on level $i+1$ result from the i-th level by partitioning. The j-th interface resulting from the partitioning on level i is denoted by $\Gamma_{i,j}$. Such recursive partitioning can be represented as a tree structure with the subdomains as leaves and the interfaces as inner nodes. The total number of subdomains in level i is denoted by s_i. We assume for simplicity of notation that all leaves belong to level l. If we consider, e.g., a two-level partitioning where every domain is recursively partitioned into two subdomains, we obtain the substructure tree 2.2(a).

Let $\mathcal{E}_{i,j}\lambda$ be the harmonic extension of a trace function $\lambda \in \Lambda_{i,j}$ onto $\Omega_{i,j}$ that vanishes outside of $\Omega_{i,j}$, where

$$\Lambda_{i,j} := \{\lambda \in H^{1/2}(\Gamma_{i,j}) \mid \lambda = v|_{\Gamma_{i,j}}, v \in H_0^1(\Omega_{i,j})\}.$$

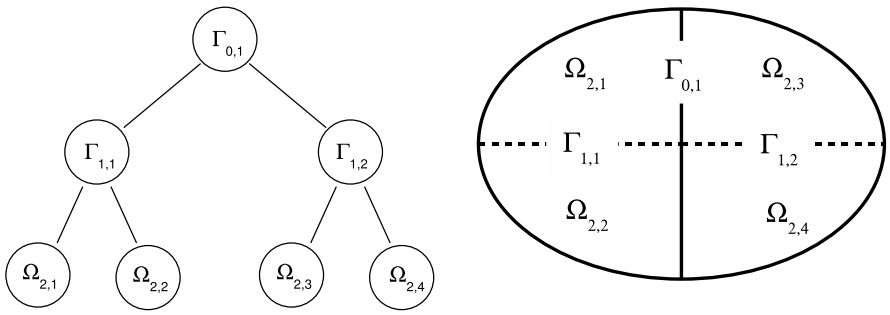
Then similarly to (2.4)-(2.5) we define

$$V_{i,j}^0 := \{v \in H_0^1(\Omega) \mid v|_{\Omega\setminus\Omega_{i,j}} = 0\}, \tag{2.17}$$

$$V_{\Gamma_{i,j}} := \{v \in H_0^1(\Omega) \mid v = \mathcal{E}_{i,j}\lambda, \lambda \in \Lambda_{i,j}\}, \tag{2.18}$$

$$V_{\Gamma_i} := \bigoplus_{j=1}^{s_i} V_{\Gamma_{i,j}} \tag{2.19}$$

By induction we obtain the following generalization of corollary 2.1 which is proven in [5].



(a) Substructure tree (b) Partitioned domain

FIG. 2.2. *Two-level partitioning*

THEOREM 2.3 ([5]). *If l is the level number for leaf subdomains in the substructure tree, then*

$$H_0^1(\Omega) = \left(\bigoplus_{j=1}^{s_l} V_{l,j}^0\right) \oplus \left(\bigoplus_{j=0}^{l-1} V_{\Gamma_j}\right) \tag{2.20}$$

is an orthogonal decomposition w.r.t the bilinear form $a_\Omega(\cdot,\cdot)$.
We obtain s_l parabolic problems of type:
Find $u \in L^2((0,T); H_0^1(\Omega_{l,j}))$, $j = 1,\dots,s_l$, s.t.

$$\begin{gathered}\frac{d}{dt}(u(t),v)_{\Omega_{l,j}} + a_{\Omega_{l,j}}(u(t),v) = (f(t),v)_{\Omega_{l,j}},\ \forall v \in H_0^1(\Omega_{l,j}) \\ u(0) = 0\end{gathered} \tag{2.21}$$

and $\sum_{j=0}^{l-1} s_j$ parabolic problems on the interfaces:
Find $\lambda \in L^2((0,T); \Lambda_{i,j})$, s.t.

$$\begin{aligned}\frac{d}{dt}\langle \mathcal{M}_{i,j}\lambda(t),\mu\rangle + \langle \mathcal{S}_{i,j}\lambda(t),\mu\rangle =& \langle \chi_{i,j}(f),\mu\rangle,\ \forall \mu \in \Lambda_{i,j} \\ \lambda(0) = 0,\ j = 1,\dots,s_i,\ i = 0,\dots,l-1&\end{aligned} \tag{2.22}$$

A semi-discretization in the multi-level case leads to a nested structure of the mass and stiffness matrices, where every diagonal block associated with a subdomain $\Omega_{i,j}$ has again arrow head structure (2.12)[1], if $\Omega_{i,j}$ is not a leaf of the substructure tree. Hence an annihilation of the off-diagonal blocks of the stiffness matrix in a bottom-up manner through the substructure tree yields a multi-level version of (2.13). The systems made of the diagonal blocks correspond to the continuous problems (2.21) and (2.22). Thus, after discretizing (2.2)-(2.3) the substructuring process consists of partitioning the state space recursively and the block-elimination described above. The subsystems are obtained by ignoring the remaining coupling blocks in the mass matrix.

[1] Maybe with more than two subdomains

3. Model reduction. Consider the linear time-invariant system

$$\begin{aligned} M\dot{x}(t) &= Ax(t) + Bu(t),\ x(0) = 0 \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3.1}$$

$A, M \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. By the Laplace transformation we obtain the transfer function

$$G(s) = C\left(sM - A\right)^{-1} B + D. \tag{3.2}$$

We will refer to (3.2) or equivalently to the system (3.1) by

$$\left[\begin{array}{c|c} sM - A & B \\ \hline C & D \end{array}\right].$$

If $M = I$, we will only give A instead of $sI - A$ in the compact notation above, i.e., $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. We can always transform the system to that case if M is nonsingular.
The idea of model reduction is to approximate system (3.1) by systems with significantly smaller state space dimension but nearly the same input-output behaviour. Usually one determines appropriate matrices $L \in \mathbb{R}^{\tilde{n}\times n}$ and $R \in \mathbb{R}^{n\times\tilde{n}}$, so that the approximation is obtained via

$$\tilde{G} = \left[\begin{array}{c|c} L(sM - A)R & LB \\ \hline RC & D \end{array}\right].$$

Additive model reduction tries to minimize the error $G - \tilde{G}$ in some sense. For example $\|G - \tilde{G}\|_{\mathcal{H}_\infty}$ should be as small as possible, because the relative output error is bounded by this term, i.e. $\|y - \tilde{y}\|_{\mathcal{H}_2} \leq \|G - \tilde{G}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{H}_2}$. More information about the Hardy spaces $\mathcal{H}_2$, $\mathcal{H}_\infty$ and their properties is to be found in [12],[17], [15] and [18].

3.1. SVD-based methods. The class of SVD-based methods is very popular due to their well-understood properties like existence of global error bounds and preservation of system characteristics. There exist quite a number of different SVD-based methods, e.g., balanced truncation, optimal Hankel norm approximation and stochastic balancing. The high computational effort of these methods is a severe disadvantage. Hence the applicability to large systems is still a challenging topic. We will describe balanced truncation as one of the most important representatives of this class.

3.1.1. Balanced truncation. Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization of a stable LTI system, i.e., $Re(\lambda) < 0$, $\lambda \in \text{spec}\{A\}$, and the dimension of the state space n is minimal. Then the two Lyapunov equations

$$A^T L_o + L_o A + CC^T = 0$$

and

$$AL_c + L_c A^T + B^T B = 0$$

have unique solutions L_o, L_c. The matrix L_o is called *observability gramian* and L_c is called *controllability gramian*. The spectrum of $L_c L_o$ is real and positive and the

square roots $\sigma_1 \geq \cdots \geq \sigma_n > 0$ of the eigenvalues are called *Hankel singular values*. The system is called *balanced* if $L_c = L_o = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$. In the case of a balanced, stable and minimal realization consider the following consistent partitioning of the matrices

$$\left[\begin{array}{cc|c} A_{1,1} & A_{1,2} & B_1 \\ A_{2,1} & A_{2,2} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right],$$

where $A_{1,1} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ is associated with the states of the $\tilde{n}$ biggest Hankel singular values $\sigma_1, \ldots, \sigma_{\tilde{n}}$ and $A_{2,2} \in \mathbb{R}^{n-\tilde{n}\times n-\tilde{n}}$ corresponds to $\sigma_{\tilde{n}+1}, \ldots, \sigma_n$. Then $\left[\begin{array}{c|c} A_{1,1} & B_1 \\ \hline C_1 & D \end{array}\right]$ is minimal, stable and balanced. Furthermore the global error bound

$$\|G - \tilde{G}\|_{\mathcal{H}_\infty} \leq 2 \sum_{i=\tilde{n}+1}^{n} \sigma_i \tag{3.3}$$

holds, where G and $\tilde{G}$ are the transfer functions of the original and the reduced system respectively. See [18] for more details about the system theoretic basics and balanced truncation.

3.2. Krylov-based methods. The Taylor expansion at s_0 of the transfer function (3.2) is given by

$$\begin{aligned} G(s) &= \sum_{i=0}^{\infty} \left(-C\left[(A - s_0M)^{-1}M\right]^i (A - s_0M)^{-1}B\right)(s - s_0)^i + D \\ &=: \sum_{i=0}^{\infty} \mu_i (s - s_0)^i \end{aligned} \tag{3.4}$$

where the coefficients μ_i are called the *moments* at s_0.
We denote the j-th *block Krylov space* of A and initial matrix B by

$$\mathcal{K}_j(A, B) := \operatorname{colspan}\{B, AB, \ldots, A^{j-1}B\}.$$

The moments can be built by elements of Krylov spaces and the following theorem holds.

THEOREM 3.1 ([10],[13]). *Let*

$$\begin{aligned} \mathcal{K}_{j_B}((A - s_0M)^{-1}M, (A - s_0M)^{-1}B) &\subset \mathit{colspan}\{R\}, \\ \mathcal{K}_{j_C}((A - s_0M)^{-H}M^H, (A - s_0M)^{-H}C^H) &\subset \mathit{colspan}\{L^H\}. \end{aligned} \tag{3.5}$$

Then

$$\mu_i = \tilde{\mu}_i, \ i = 0, \ldots, j_B + j_C - 1,$$

where μ_i and $\tilde{\mu}_i$ are the moments of G and $\tilde{G}$ respectively.
Hence, we can match an arbitrary number of moments by fulfilling condition (3.5). To this end we have to compute bases of the Krylov spaces by appropriate algorithms, e.g., Arnoldi or Rational Krylov. Then it holds

$$G - \tilde{G} = \mathcal{O}((s - s_0)^{j_B + j_C}).$$

4. Reduction algorithm for LTI systems. The substructuring strategy presented in section 2 motivates an algorithm for general LTI systems, even if they are not derived from problems like (2.1).

The reduction algorithm consists basically of two phases. First the system is transformed and divided into several subsystems. In the second phase the subsystems are reduced.

4.1. Substructuring. Consider the realization $\left[\begin{array}{c|c} sM-A & B \\ \hline C & \end{array}\right]$ of an LTI system (3.1). The substructuring of the LTI system is analog to the procedure in 2.

Algorithm 1 Substructuring of LTI systems

1: Partition the state space hierarchically

2: Transform the LTI system by regular matrices U_l, U_r, s.t.

$$A^{(1)} := U_l A U_r,\ M^{(1)} := U_l M U_r,\ B^{(1)} := U_l B,\ C^{(1)} := C U_r$$

and $A^{(1)} = \mathrm{diag}(A^{(1)}_{1,1}, \ldots, A^{(1)}_{s,s})$ is block-diagonal

3: Define new subsystems $\left[\begin{array}{c|c} sM^{(1)}_{i,i} - A^{(1)}_{i,i} & B^{(1)}_i \\ \hline C^{(1)}_i & \end{array}\right]$, $i = 1, \ldots, s$ induced by the diagonal blocks

Note that the transformation matrices U_l and U_r are never built explicitly. The transformation is simply done by eleminating the off-diagonal blocks in A. The partitioning due to the sparsity structure of the matrix A can be performed by a graph partitioner like Metis[2].

We illustrate algorithm 1 by an example. Consider again a two-level partitioning of the state space. On each level we divide every domain in two parts. We obtain four subdomains, one interface on the first level and two interfaces on the second level. The system matrices have the following structure after possible permutations:

$$A = \begin{pmatrix} A_{1,1} & & A_{1,3} & & & & A_{1,7} \\ & A_{2,2} & A_{2,3} & & & & A_{2,7} \\ A_{3,1} & A_{3,2} & A_{3,3} & & & & A_{3,7} \\ & & & A_{4,4} & & A_{4,6} & A_{4,7} \\ & & & & A_{5,5} & A_{5,6} & A_{5,7} \\ & & & A_{6,4} & A_{6,5} & A_{6,6} & A_{6,7} \\ A_{7,1} & A_{7,2} & A_{7,3} & A_{7,4} & A_{7,5} & A_{7,6} & A_{7,7} \end{pmatrix},$$

$$M = \begin{pmatrix} M_{1,1} & & M_{1,3} & & & & M_{1,7} \\ & M_{2,2} & M_{2,3} & & & & M_{2,7} \\ M_{3,1} & M_{3,2} & M_{3,3} & & & & M_{3,7} \\ & & & M_{4,4} & & M_{4,6} & M_{4,7} \\ & & & & M_{5,5} & M_{5,6} & M_{5,7} \\ & & & M_{6,4} & M_{6,5} & M_{6,6} & M_{6,7} \\ M_{7,1} & M_{7,2} & M_{7,3} & M_{7,4} & M_{7,5} & M_{7,6} & M_{7,7} \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \\ B_7 \end{pmatrix}, \tag{4.1}$$

$$C = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 \end{pmatrix}.$$

[2] http://www-users.cs.umn.edu/~karypis/metis/metis/index.html

The transformation of the system by annihilating the off-diagonal blocks in A yields

$$A^{(1)} = \operatorname{diag}\left(A_{1,1}, A_{2,2}, A_{3,3}^{(1)}, A_{4,4}, A_{5,5}, A_{6,6}^{(1)}, A_{7,7}^{(1)}\right),$$

$$M^{(1)} = \begin{pmatrix} M_{1,1} & & M_{1,3}^{(1)} & & & & M_{1,7}^{(1)} \\ & M_{2,2} & M_{2,3}^{(1)} & & & & M_{2,7}^{(1)} \\ M_{3,1}^{(1)} & M_{3,2}^{(1)} & M_{3,3}^{(1)} & & & & M_{3,7}^{(1)} \\ & & & M_{4,4} & & M_{4,6}^{(1)} & M_{4,7}^{(1)} \\ & & & & M_{5,5} & M_{5,6}^{(1)} & M_{5,7}^{(1)} \\ & & & M_{6,4}^{(1)} & M_{6,5}^{(1)} & M_{6,6}^{(1)} & M_{6,7}^{(1)} \\ M_{7,1}^{(1)} & M_{7,2}^{(1)} & M_{7,3}^{(1)} & M_{7,4}^{(1)} & M_{7,5}^{(1)} & M_{7,6}^{(1)} & M_{7,7}^{(1)} \end{pmatrix}, \; B^{(1)} = \begin{pmatrix} B_1 \\ B_2 \\ B_3^{(1)} \\ B_4 \\ B_5 \\ B_6^{(1)} \\ B_7^{(1)} \end{pmatrix},$$

$$C^{(1)} = \begin{pmatrix} C_1 & C_2 & C_3^{(1)} & C_4 & C_5 & C_6^{(1)} & C_7^{(1)} \end{pmatrix}.$$

Note that only blocks associated with interfaces are changed by this transformation. By ignoring the coupling blocks in $M^{(1)}$ we end up with the subsystems

$$\left[\begin{array}{c|c} sM_{i,i} - A_{i,i} & B_i \\ \hline C_i & \end{array}\right], \quad i \in \{1, 2, 4, 5\}$$

corresponding to subdomains and the interface subsystems

$$\left[\begin{array}{c|c} sM_{i,i}^{(1)} - A_{i,i}^{(1)} & B_i^{(1)} \\ \hline C_i^{(1)} & \end{array}\right], \quad i \in \{3, 6, 7\}.$$

4.2. Reduction. Instead of reducing the original system we reduce the subsystems arising from the substructuring phase.

Algorithm 2 Reduction of LTI systems

1: Reduce the subsystems (subdomains and interfaces) by model reduction techniques (especially SVD-based methods), i.e., determine appropriate $L_i^{(2)}$, $R_i^{(2)}$, s.t.

$$\left[\begin{array}{c|c} s\tilde{M}_{i,i} - \tilde{A}_{i,i} & \tilde{B}_i \\ \hline \tilde{C}_i & \end{array}\right] = \left[\begin{array}{c|c} L_i^{(2)}\left(sM_{i,i}^{(1)} - A_{i,i}^{(1)}\right)R_i^{(2)} & L_i^{(2)}B_i^{(1)} \\ \hline C_i^{(1)}R_i^{(2)} & \end{array}\right].$$

2: Project the coupling blocks, i.e.,

$$\tilde{M}_{i,j} = L_i^{(2)} M_{i,j}^{(1)} R_j^{(2)}, \; i \neq j.$$

The reduced system

$$\begin{aligned} \tilde{M}\dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \; \tilde{x}(0) = 0 \\ \tilde{y}(t) &= \tilde{C}\tilde{x}(t) \end{aligned} \tag{4.2}$$

is of order $\sum_{i=1}^{s} \tilde{n}_i$, where $\tilde{n}_i$ is the order of the i-th reduced subsystem.

The original AMLS method applies the substructuring steps to a generalized eigenvalue problem and the resulting subproblems are reduced by modal truncation.

In contrast we utilize the advantages of model reduction techniques and do not ignore the input and output behaviour of the system. Another advantage of the combination of hierarchical substructuring and model reduction is the possibility to use SVD-based techniques for the reduction of large systems because they are only applied to smaller subsystems. Furthermore the hierarchical substructuring approach yields smaller interfaces than single level methods like CMS.

5. Error analysis. Without loss of generality we only consider systems in the frequency domain of type

$$\begin{aligned} s\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y = y_1 + y_2 &= \begin{pmatrix} C_1 & C_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \end{aligned} \tag{5.1}$$

since the transformation of the LTI system in the substructuring phase doesn't change the system and yields a feasible realization. Furthermore, it is sufficient to restrict to a single-level partitioning with two substructures, because one substructure may consist of several subdomains and interfaces and the results from this section can be applied recursively.

Let $(i,j) \in \{(1,2),(2,1)\}$. Solving system (5.1) for x_i yields

$$x_i = H_i u - H_{ij}\left(A_j x_j + B_j u\right) + H_{ij} M_{j,i} M_{i,i}^{-1}\left(A_i x_i + B_i u\right), \tag{5.2}$$

where we use the definitions

$$H_i := (sM_{i,i} - A_i)^{-1} B_i,\ H_{ij} := (sM_{i,i} - A_i)^{-1} M_{i,j} (M_{j,j}^S)^{-1},\ H_{glob} := (sM - A)^{-1} B \tag{5.3}$$

and $M_{j,j}^S := M_{j,j} - M_{j,i} M_{i,i}^{-1} M_{i,j}$.
We define the projections

$$P_i := R_i (L_i (sM_{i,i} - A_i) R_i)^{-1} L_i (sM_{i,i} - A_i) \tag{5.4}$$

$$Q_i := M_{i,i} R_i (L_i M_{i,i} R_i)^{-1} L_i. \tag{5.5}$$

onto colspan$\{R_i\}$ and rowspan$\{L_i\}$, respectively.
Similarly to (5.2) it holds

$$R_i \tilde{x}_i = \tilde{H}_i u - \tilde{H}_{ij}\left(A_j R_j \tilde{x}_j + B_j u\right) + \tilde{H}_{ij} M_{j,i} M_{i,i}^{-1} Q_i \left(A_i R_i \tilde{x}_i + B_i u\right) \tag{5.6}$$

with

$$\tilde{H}_i := R_i (s\tilde{M}_{i,i} - \tilde{A}_i)^{-1} \tilde{B}_i, \quad \tilde{H}_{ij} := R_i (s\tilde{M}_{i,i} - \tilde{A}_i)^{-1} \tilde{M}_{i,j} (\tilde{M}_{j,j}^S)^{-1} L_j \tag{5.7}$$

and $\tilde{M}_{j,j}^S := \tilde{M}_{j,j} - \tilde{M}_{j,i} \tilde{M}_{i,i}^{-1} \tilde{M}_{i,j}$. By subtracting (5.6) from (5.2) we finally obtain the system of equations

$$\begin{aligned} \left[I - \tilde{H}_{ij} M_{j,i} M_{i,i}^{-1} Q_i A_i\right]\left(x_i - R_i \tilde{x}_i\right) + \tilde{H}_{ij} A_j \left(x_j - R_j \tilde{x}_j\right) &= \Psi_i, \\ (i,j) \in \{(1,2),(2,1)\}, & \end{aligned}$$

where

$$\begin{aligned} \Psi_i := &\left(H_i - \tilde{H}_i\right) u \\ &- \left(H_{ij} - \tilde{H}_{ij}\right)\left(A_j x_j + B_j u\right) \\ &+ \left(H_{ij} M_{j,i} M_{i,i}^{-1} - \tilde{H}_{ij} M_{j,i} M_{i,i}^{-1} Q_i\right)\left(A_i x_i + B_i u\right). \end{aligned} \tag{5.8}$$

This system yields

$$x_i - R_i\tilde{x}_i = \left[I - \tilde{H}_{ij}M_{j,i}M_{i,i}^{-1}Q_iA_i - \tilde{H}_{ij}A_j\left(I - \tilde{H}_{ji}M_{i,j}M_{j,j}^{-1}Q_jA_j\right)^{-1}\tilde{H}_{ji}A_i\right]^{-1} \cdot \left[\Psi_i - \tilde{H}_{ij}A_j\left(I - \tilde{H}_{ji}M_{i,j}M_{j,j}^{-1}Q_jA_j\right)^{-1}\Psi_j\right]. \tag{5.9}$$

Note that the difference $x_i - R_i\tilde{x}_i$ is a rational function which simplifies for $s = 0$ to

$$(x_i - R_i\tilde{x}_i)(0) = \left(\left(H_i - \tilde{H}_i\right)u\right)(0).$$

If the original and approximating subsystems are stable and minimal, $H_i - \tilde{H}_i$ and therefore $x_i - R_i\tilde{x}_i$ have no pole at $s = 0$.
The output error of the i-th subsystem is given by

$$y_i - \tilde{y}_i = C_i(x_i - R_i\tilde{x}_i). \tag{5.10}$$

We simplify (5.9) by neglecting all Terms in (5.9) and (5.8) arising from feedback loops from a subsystem to itself, because we assume that the main error source is the mutual influence of one system on the other. Hence we obtain

$$y_i - \tilde{y}_i \approx C_i\hat{\Psi}_i - C_i\tilde{H}_{ij}A_j\hat{\Psi}_j, \tag{5.11}$$

with

$$\begin{aligned}\hat{\Psi}_i := &\left(H_i - \tilde{H}_i\right)u \\ &- \left(H_{ij} - \tilde{H}_{ij}\right)(A_jx_j + B_ju).\end{aligned} \tag{5.12}$$

THEOREM 5.1. *The right-hand side of* (5.11) *can be bounded by*

$$\frac{\|C_i\hat{\Psi}_i - C_i\tilde{H}_{ij}A_j\hat{\Psi}_j\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_2}} \leq \|G_i - \tilde{G}_i\|_{\mathcal{H}_\infty} + \Theta_1 + \Theta_2, \tag{5.13}$$

where G_i and $\tilde{G}_i$ are the transfer functions of the i-th subsystem and its approximation respectively and

$$\Theta_1 = \|C_iH_{ij} - C_i\tilde{H}_{ij}\|_{\mathcal{H}_\infty}\left(\|A_j\|_2\|H_{glob}\|_{\mathcal{H}_\infty} + \|B_j\|_2\right)$$
$$\Theta_2 = \|C_i\tilde{H}_{ij}A_j\|_{\mathcal{H}_\infty}\left[\|H_j - \tilde{H}_j\|_{\mathcal{H}_\infty} + \|H_{ji} - \tilde{H}_{ji}\|_{\mathcal{H}_\infty}\left(\|A_i\|_2\|H_{glob}\|_{\mathcal{H}_\infty} + \|B_i\|_2\right)\right].$$

Proof. The $\mathcal{H}_2$ norm induces the $\mathcal{H}_\infty$ norm, i,e.

$$\|G\|_{\mathcal{H}_\infty} = \sup_{u\in\mathcal{H}_2, u\neq 0}\|Gu\|_{\mathcal{H}_2}/\|u\|_{\mathcal{H}_2}.$$

Furthermore the inequality

$$\|x_i\|_{\mathcal{H}_2} \leq \|x\|_{\mathcal{H}_2} = \|H_{glob}u\|_{\mathcal{H}_2} \leq \|H_{glob}\|_{\mathcal{H}_\infty}\|u\|_{\mathcal{H}_2}$$

holds.
Hence the inequality (5.13) follows immediately from (5.12) and the triangle inequality. □

Remarks: The inequality (5.13) can be used as error indicator, because it holds

$$\|G - \tilde{G}\|_{\mathcal{H}_\infty} \leq \sum_{i=1}^{2} \sup_{u \in \mathcal{H}_2, u \neq 0} \frac{\|y_i - \tilde{y}_i\|_{\mathcal{H}_2}}{\|u\|_{\mathcal{H}_2}}.$$

It should be noted that the first term of the right-hand side of (5.13) is usually very small due to the properties of SVD-based reduction methods. Θ_1 and Θ_2 are additive error terms arising from the coupling in M. Note that Θ_1 and Θ_2 may be relatively large, although $\|G_i - \tilde{G}_i\|_{\mathcal{H}_\infty}$ is small, because we ignore the coupling blocks in $M^{(1)}$ during the reduction of the subsystems. Thus, we get good approximations of the transfer functions G_i, but the approximations (5.7) of the transfer functions (5.3) that describe the coupling behaviour may be worse. The worst case is that we have to keep basically the entire state space to obtain a good approximation of the coupled system. Nevertheless the method performs very well in practical applications.

If we can make further assumptions on the coupling blocks, the reduced system (4.2) is a Padé-type approximation.

LEMMA 5.2. *Let*

$$\textit{rowspan}\{M_{j,i}M_{i,i}^{-1}\} \subset \textit{rowspan}\{L_i\}, \tag{5.14}$$

hold for $(i,j) \in \{(1,2),(2,1)\}$*, then*

$$\tilde{M}_{i,j}(\tilde{M}_{j,j}^S)^{-1}L_j = L_i M_{i,j}(M_{j,j}^S)^{-1} \tag{5.15}$$

Proof. By the Sherman-Morrison-Woodbury formula

$$\begin{aligned}
\tilde{M}_{i,j}(\tilde{M}_{j,j}^S)^{-1}L_j &= \tilde{M}_{i,j}\left(\tilde{M}_{j,j} - \tilde{M}_{j,i}\tilde{M}_{i,i}^{-1}\tilde{M}_{i,j}\right)^{-1} L_j \\
&= \tilde{M}_{i,j}\left(\tilde{M}_{j,j} - L_j M_{j,i}M_{i,i}^{-1}Q_i M_{i,j}R_j\right)^{-1} L_j \\
&= \tilde{M}_{i,j}\tilde{M}_{j,j}^{-1}L_j + \tilde{M}_{i,j}\tilde{M}_{j,j}^{-1}L_j M_{j,i}\left(M_{i,i} - Q_i M_{i,j}R_j\tilde{M}_{j,j}^{-1}L_j M_{j,i}\right)^{-1} \\
&\quad \cdot Q_i M_{i,j}R_j\tilde{M}_{j,j}^{-1}L_j \\
&= L_i M_{i,j}M_{j,j}^{-1}Q_j + L_i M_{i,j}M_{j,j}^{-1}Q_j M_{j,i}\left(M_{i,i} - Q_i M_{i,j}M_{j,j}^{-1}Q_j M_{j,i}\right)^{-1} \\
&\quad \cdot Q_i M_{i,j}M_{j,j}^{-1}Q_j.
\end{aligned}$$

By (5.14) it follows

$$\begin{aligned}
\tilde{M}_{i,j}(\tilde{M}_{j,j}^S)^{-1}L_j &= L_i M_{i,j}M_{j,j}^{-1} + L_i M_{i,j}M_{j,j}^{-1}M_{j,i}\left(M_{i,i} - Q_i M_{i,j}M_{j,j}^{-1}M_{j,i}\right)^{-1} Q_i M_{i,j}M_{j,j}^{-1} \\
&= L_i M_{i,j}\left(M_{j,j} - M_{j,i}M_{i,i}^{-1}Q_i M_{i,j}\right)^{-1} = L_i M_{i,j}\left(M_{j,j} - M_{j,i}M_{i,i}^{-1}M_{i,j}\right)^{-1} \\
&= L_i M_{i,j}(M_{j,j}^S)^{-1}.
\end{aligned}$$

□

The next lemma examines the differences of the functions (5.3) and (5.7).

LEMMA 5.3. *If* (5.14) *holds, then*

1. $H_{ij} - \tilde{H}_{ij} = (I - P_i)H_{ij}$
2. $H_i - \tilde{H}_i = (I - P_i)H_i$.

Let

$$\mathcal{K}_l\left(A_i^{-1}M_{i,i}, A_i^{-1}[M_{i,j}, B_i]\right) \subset \mathit{colspan}\{R_i\}, \tag{5.16}$$

hold additionally, then even

1. $H_{ij} - \tilde{H}_{ij} = \mathcal{O}(s^l)$
2. $H_i - \tilde{H}_i = \mathcal{O}(s^l)$

follows.

Proof. We consider only the function H_{ij} because the proof for H_i is analog. The first statement follows directly from lemma 5.2, because

$$\begin{aligned}\tilde{H}_{ij} &= R_i(s\tilde{M}_{i,i} - \tilde{A}_i)^{-1}\tilde{M}_{i,j}(\tilde{M}_{j,j}^S)^{-1}L_j = R_i(s\tilde{M}_{i,i} - \tilde{A}_i)^{-1}L_iM_{i,j}(M_{j,j}^S)^{-1} \\ &= P_i(sM_{i,i} - A_i)^{-1}M_{i,j}(M_{j,j}^S)^{-1}.\end{aligned}$$

The transfer function $P_iH_{ij} = R_i(s\tilde{M}_{i,i} - \tilde{A}_i)^{-1}L_iM_{i,j}(M_{j,j}^S)^{-1}$ is obtained by reducing the order of H_{ij} by the matrices L_i and R_i. Hence the second part is a consequence of the moment matching theorem 3.1 from model reduction by means of Krylov spaces (see [13]). □

We obtain the following theorem.

THEOREM 5.4. *If zero is not a pole of* (5.9) *and* (5.14), (5.16) *are fulfilled, then*

$$y_i - \tilde{y}_i = G_i - \tilde{G}_i + \mathcal{O}(s^l). \tag{5.17}$$

Proof. Follows from lemma 5.3 and equations (5.8), (5.9). □

Hence, the additive error resulting from the coupling (compare theorem 5.1) can be influenced by fulfilling (5.14) and (5.16). To this end bases of the Krylov spaces have to be computed by an appropriate algorithm like Arnoldi and the projection spaces have to be extended by those bases. But this strategy may turn out to be impracticable, because multi-level variants could lead to too many coupling blocks that would have to be considered. Perhaps partial realizations by a selection of initial vectors from coupling blocks will improve the global approximation of the system. In [3] Bekas and Saad make similar proposals for AMLS.

6. Numerical results. We apply our method to two examples taken from the Oberwolfach Benchmark Collection[3].

6.1. Cooling of steel profiles. The cooling process of steel profiles in a rolling mill requires different temperatures of the material. A discretization of the arising optimization problem yields a state space model of order $n = 20209$ with seven inputs and six outputs. By decomposing every domain recursively into five subdomains we obtain in the second level 25 subdomains and 6 interfaces. The 31 subsystems are reduced by balancing free truncation with absolute error tolerance 10^{-3}. We obtain an approximation of order 48. The magnitude of the frequency responses from the 6th input to the 2nd output are shown in 6.1 and the error is shown in 6.2.

6.2. Tunable optical filter. This benchmark is obtained by the development of an optical filter, which is tunable by thermal means. The discretization of a simplified 3D model yields an LTI system with state space dimension 106437, one input and five outputs. By recursive partitioning of a domain in two subdomains up to level 7 we

[3] http://www.imtek.uni-freiburg.de/simulation/benchmark

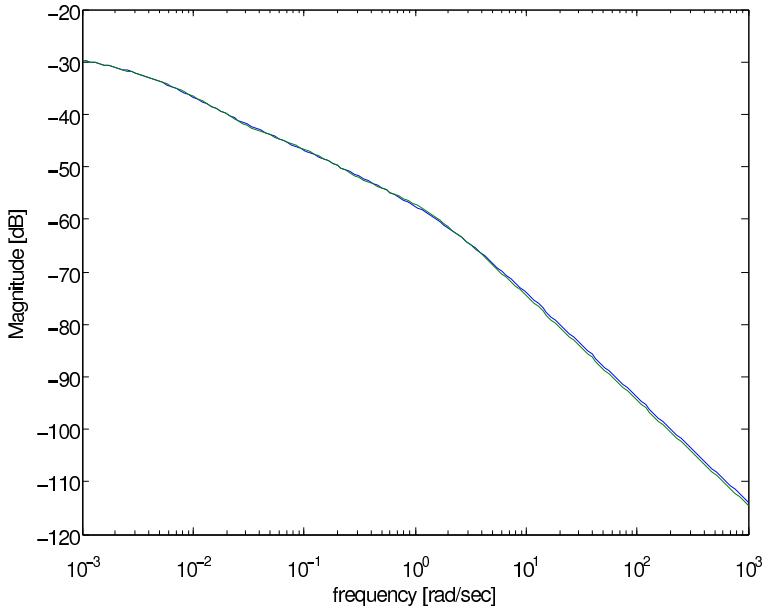


FIG. 6.1. *Magnitude of the frequency responses (original and approximation) of the cooling problem in [dB]*

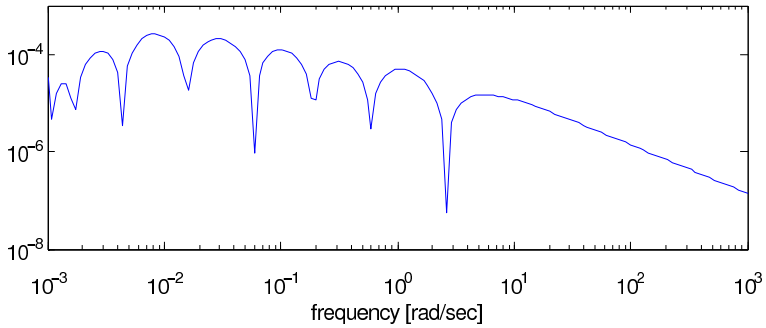


FIG. 6.2. *Absolute error of the approximation of the cooling problem*

obtain 255 subsystems, where 127 subsystems belong to interfaces and 128 subsystems correspond to subdomains. Reduction of the subsystems by balancing free truncation with error bound 10^{-2} yields a reduced system of order 16. Figures 6.3 and 6.4 illustrate the magnitude of the frequency responses from ouput one and the error.

7. Conclusions. We have shown that the combination of hierarchical substructuring and model reduction methods is an efficient approach to make SVD-based methods applicable even to very large systems. The use of model reduction methods instead of modal truncation takes the input and output behaviour of the system into account and yields smaller reduced subsystems. Furthermore we can make use of the advantages of SVD-based reduction methods on the substructure level.

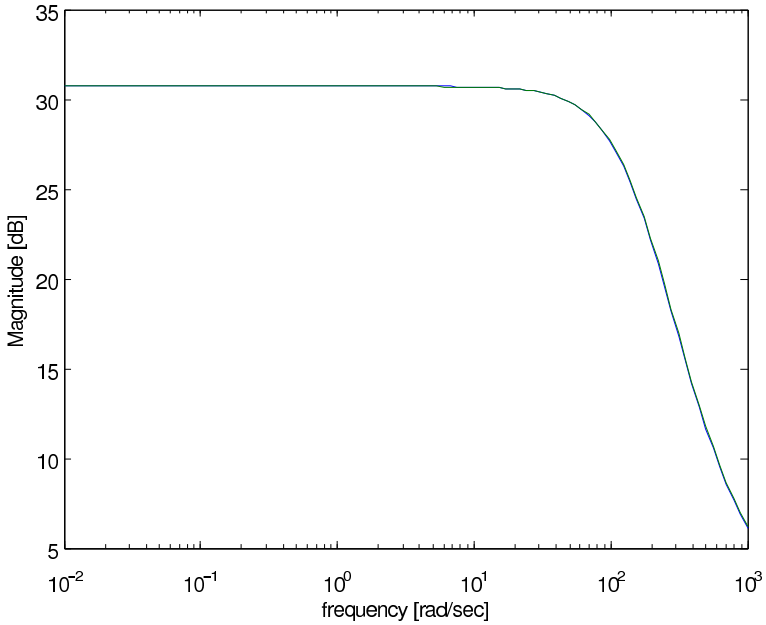


FIG. 6.3. *Magnitude of the frequency responses (original and approximation) of the filter benchmark in [dB]*

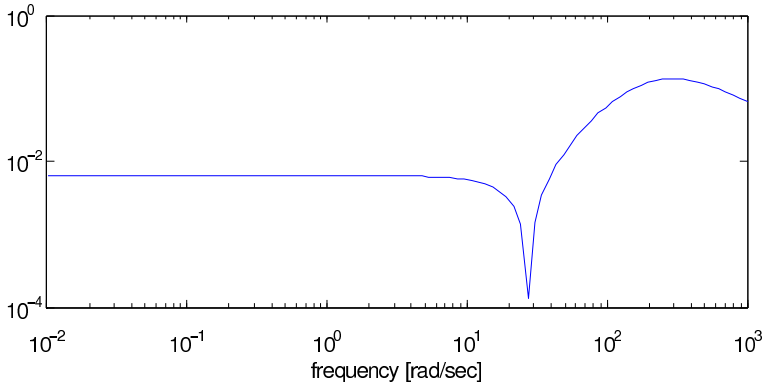


FIG. 6.4. *Absolute error of the approximation of the filter benchmark*

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