Observability of non-autonomous systems

1) Linear Cauchy Problems
2) Well-posedness of Cauchy Problems
3) Control Systems and Observability

} Vorbereitung

Evolution (through time) of a system (and control thereof)

2 independent variables: Time $t \in J \subseteq \mathbb{R}$, System state $x \in X$

statespace

Assumption: The dynamics of the system depends only on the rate of change of the state

→ (abstract) ODE

$$\left.\begin{aligned} \dot{x}(t) &= F(t, x(t)) \\ x(0) &= x_0 \end{aligned}\right\} \text{"initial value problem"}$$

Wanted: $X: \mathbb{R}_+ \to X$ for which we can define a derivative

(X needs to be at least TVS with T_2)

Standard: X Banach space

Usually X Lebesgue or Hölder
Sobolev

Example: Diffusion equation $\mathbb{R}^d$

$$\dot{x} = -\sum_{i=1}^{d} \partial_{ii} x \xrightarrow{\text{Solution}} \underbrace{S(t)x_0 = \overset{\text{Gauß}}{\mu_t} * x_0}_{\text{"Diffusion semigroup"}}$$

$$x(0) = x_0 \in L^2(\mathbb{R}^d)$$

$$\dot{x} = -Ax \quad (\text{with } A = \Delta)$$

abstract Cauchy-Problem

Usually A is a closed densely operator with domain $D(A)$

2) Well-posedness of Cauchy-Problems

When do we have a meaningful model?

I Existence

II Uniqueness   of a solution

III Corinous dependence on "the data"

Cauchy-Problem (CP) is called well-posed if

(i) $\forall x \in D(A)$ there exists a unique solution to CP

(ii) $(x_n) \subset D(A)$, $x_n \to 0$ in X

then $x(t, x_n) \to 0$

(uniformly on compact sets)

Well-posedness leads to a solution operator:

$S : \mathbb{R}_+ \to L(X)$ such

$x(t) := S(t)x_0$ is classical to (CP)

(S is a homomorphisms of semigroups

$$S(t+s) = S(t) \overset{\text{concatenation}}{\circ} S(s)$$

3) Control systems and observability

$$\text{(CP)} \quad \dot{x} = -Ax, \quad t \in (0,T], \quad x(0) = x_0 \in X$$

$$y(t) = Cx(t), \quad C \in \mathcal{L}(X,Y)$$

Suppose that (CP) is well-posed and $(S(t))$ the corresponding semigroup.

Can we recover $\underbrace{\text{"Information"}}_{\|S(T)x_0\|_{L^2}}$ about the state of the $\underbrace{\text{system at } t=T}_{S(T)x_0}$

Assume that C is not injective

Wanted: Final state observability estimate

$$\|S(T)x_0\|_X \leq C_{obs} \cdot \underbrace{\|y(t)\|_{L^r((0,T),Y)}}_{\left(\int_0^T \|y(t)\|_Y^r \, dt\right)^{1/r}}$$

Dennis, Christian, Martin T. (GST)

2 Ingredients:

1) Uncertainty relation $\| P_\lambda x \| \leq C_{uncert} \| C P_\lambda x \|$

2) Dissipation estimate $\| (Id - P_\lambda) S_t x \| \leq C_{diss} e^{-c \lambda t} \|x\|$

In particular: $P_\lambda \to Id$ strongly, exponential decay

Example (Diffusion equation) $A = \pm \Delta$ on $S(\mathbb{R}^d)$

$\dot{x} = -\Delta_p x \quad t \in (0, T], \ x(0) = x \in L^p$

$y(t) = \mathbb{1}_\Omega x(t)$ Restriction operator

GST: If Ω is "thick"

$$|\Omega \cap I| \geq \rho L$$

Consider $\Omega = \bigcup_{n \in \mathbb{Z}} [2n, 2n+1] \subseteq \mathbb{R}$

GST: Thickness is sufficient for observability

GST: — " — is also necessary for observability

Non-autonomous system:

$$(naCP) \qquad \dot{x} = -A(t)x, \quad t \in [0,T], \quad x(0) = x_0 \in X$$

$$y(t) = C(t)\,x(t) \qquad \|C(t)\| \text{ is measurable bounded}$$

Well-posedness of (naCP)?

can be translated into the existence of a sol. op.

$U(t,0)x_0 = x(t)$ (Evolution Family)

$U(t,r)\,U(r,s) = U(t,s)$

Well-posedness does not fall out of the sky

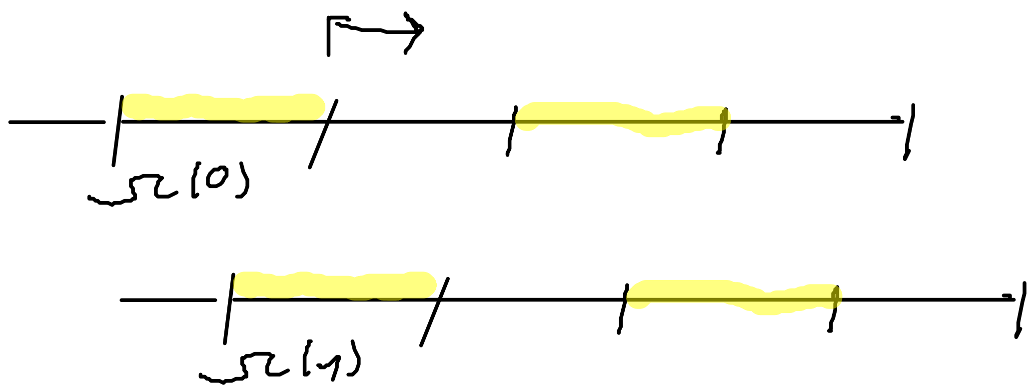
$(A(t))_{t \in [0,T]}$ (Holder stetig)

Example

$$A(t) = \sum_{ij} a_{ij}(t)\,\partial_{ij} \quad \text{on } \mathcal{S}(\mathbb{R}^d)$$

$ij \to \alpha$, $|\alpha| = 2n$

(naCP) $$\dot{x} = -A_p(t)\,x(t)$$
$$y = \mathbb{1}_{\Omega(t)}\,x(t)$$



$\Omega(t)$ thick $\Rightarrow$ Observability estimate

Observability $\Rightarrow$ Ω thick $\Omega(t) \equiv \Omega$

What about ~~oth~~ general $\Omega(t)$

It turns out that following does a good job

C^p

$\Omega(t)$ is $\{$ 'C^1', 'mean', 'L^1-mean' $\}$-thick

$$:\Longleftrightarrow \quad \frac{1}{T}\int_0^T \underbrace{|\Omega(t)\cap I|^p}_{\geq \rho L} \geq \rho L$$

$$\frac{1}{T}\int_0^T \rho L \, dt = \rho L$$

observability $\Rightarrow$ $(\Omega(t))$ C^1-mean-thick

Questions

Relax regularity of $A(t)$ (the coefficients)

maybe continuous (measurable)

- Relax reg. domains of Def. on $A(t)$ $\quad D(A(t)) \equiv D$

op norm

$$D(A(t)) = D, \quad A(\cdot) \in C^{\alpha}([0,T], \mathcal{L}(D, X))$$

$$\Updownarrow$$

$$a_{ij} \in C^{\alpha}([0,T])$$