# About myself, a solution algorithm for the Navier-Stokes Equations and the Stokes Resolvent Problem

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Institut für Mathematik
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# Contents

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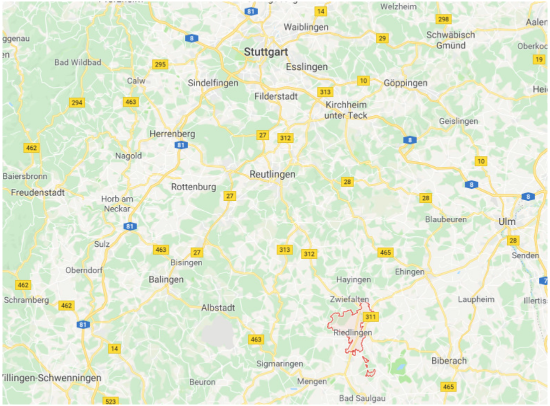

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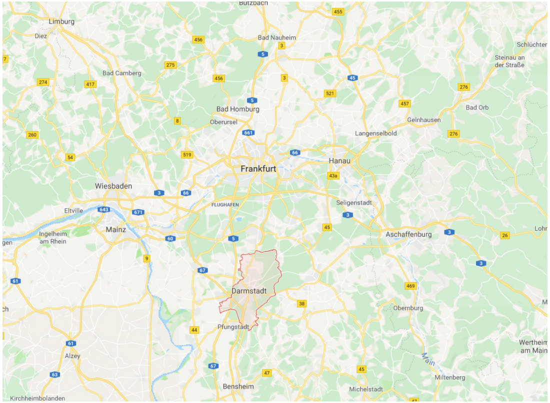

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# Research at TU Darmstadt

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- Math. Modelling and Analysis (numerical investigation of contact line movement, Prof. Bothe)

# Implementation and Performance Analyses of a Highly Efficient Algorithm for Pressure Velocity Coupling

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- Results with little waiting time
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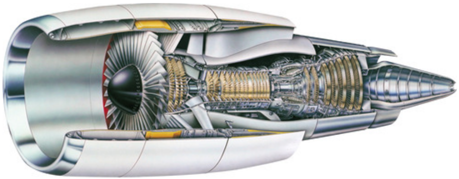

Figure: Jet turbine (GE)

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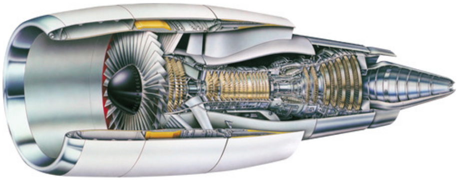

Figure: Jet turbine (GE)

Requirements

- Adaptivity
- Coupling of variables
- Scalability (parallel computations)

# Physical Model - Navier-Stokes Equations + Scalar Transport

$$a_P^{u_i} u_{P,i} + \sum_{F \in NB(P)} a_F^{u_i} u_{F,i} + \underbrace{a_P^{u_i,p} p_P + \sum_{F \in NB(P)} a_F^{u_i,p} p_F}_{\text{Pressure-velocity coupling}} + \underbrace{a_P^{u_i,T} T_P}_{\text{Boussinesq approximation}} = b_{P,u_i} \quad i = 1, \ldots, 3$$

$$a_P^p p_P + \sum_{F \in NB(P)} a_F^p p_F + \underbrace{\sum_{j=1}^{3} \left( a_P^{p,u_j} u_{P,j} + \sum_{F \in NB(P)} a_F^{p,u_j} u_{F,j} \right)}_{\text{Pressure-velocity coupling}} = b_{P,p}$$

$$a_P^T T_P + \sum_{F \in NB(P)} a_F^T T_F + \underbrace{\sum_{j=1}^{3} \left( a_P^{T,u_j} u_{P,j} + \sum_{F \in NB(P)} a_F^{T,u_j} u_{F,j} \right) + a_P^{T,p} p_P + \sum_{F \in NB(P)} a_F^{T,p} p_F}_{\text{Newton-Raphson linearization}} = b_{P,T}$$

# The PETSc Framework

A freely available and supported research code for the parallel solution of nonlinear algebraic equations

- Usable from C, C++, FORTRAN 77/90, Python and MATLAB
- Portable to any parallel system supporting MPI
- $10^{12}$ unknowns, full-machine scalability on Top-10 systems (see top500.org)

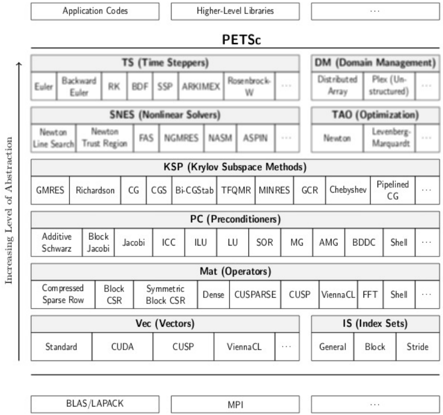


Figure: Numerical libraries of PETSc

# Adaptivity and Block Structured Grids

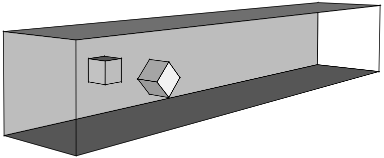

Figure: Domain of channel flow

- Cubes as obstacles
- laminar flow
- nontrivial block transitions

Figure: Blocking around obstacles

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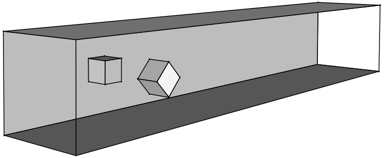

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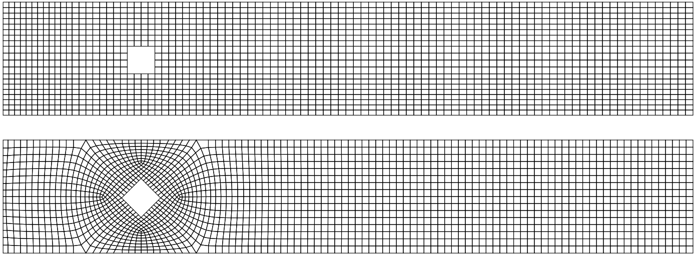

Figure: Numerical grid on west- and east boundary.

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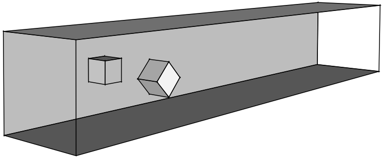

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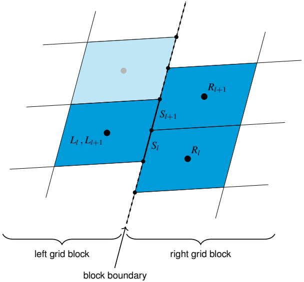
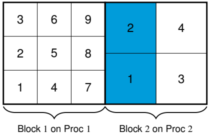
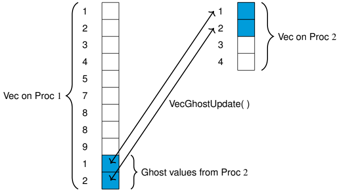


# Matrix Structure and Variable Coupling

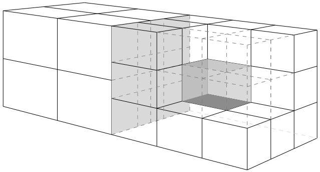
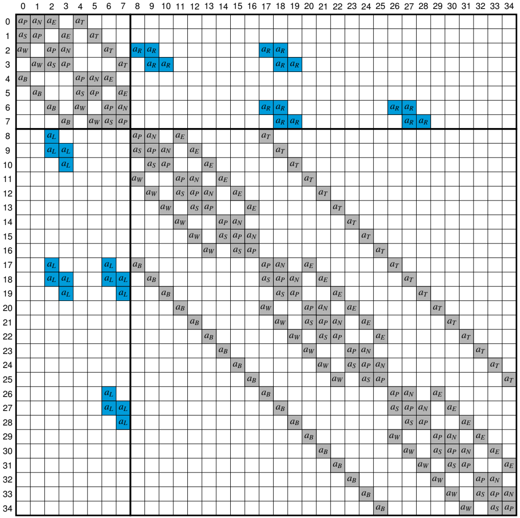


Figure: block structured grid and corresponding matrix structure (no variable coupling)



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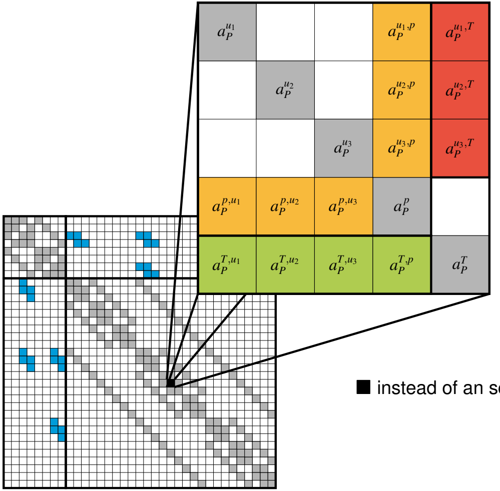


unterschiedliche Kopplungsterme:

- Pressure-Velocity
- Velocity
- Temperature-Velocity/Pressure

■ instead of an scalar entry: dense $5x5$ matrix

# MIT Benchmark - Velocity-Pressure-Temperature Coupling

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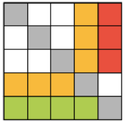
| Resolution | Solver configuration | Time s | No. of nonlinear its. |
|---|---|---|---|
| 32x32x32 | SEG | 0.3719E+02 | 203 |
| | CPLD | 0.6861E+02 | 62 |
| | TCPLD | 0.1012E+03 | 31 |
| | NRCPLD | 0.2153E+02 | 22 |
| 64x64x64 | SEG | 0.1997E+04 | 804 |
| | CPLD | 0.7687E+03 | 63 |
| | TCPLD | 0.1278E+04 | 59 |
| | NRCPLD | 0.4240E+03 | 17 |
| 128x128x128 | SEG | 0.5197E+05 | 3060 |
| | CPLD | 0.1860E+05 | 74 |
| | TCPLD | 0.1950E+05 | 50 |
| | NRCPLD | 0.6155E+04 | 18 |



# Existence of Solutions to the Navier-Stokes Equations - Agenda

Navier-Stokes equations for homogeneous, incompressible fluids:

$$\begin{aligned} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla \pi &= f \quad && t \in (0,T), x \in \Omega \\ \nabla \cdot u &= 0 \quad && t \in (0,T), x \in \Omega \\ u(0) &= a \quad && x \in \Omega \\ u|_{\partial\Omega} &= 0 \quad && t \in (0,T). \end{aligned}$$

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1. Construct operator $A\colon \mathrm{D}(A) \to X$, $X$ suitable Banach space such that
$$u \in \mathrm{D}(A) \quad \Longleftrightarrow \quad \nabla \cdot u = 0,\ u|_{\partial\Omega} = 0 \text{ and } \exists \pi \text{ with } Au = -\Delta u + \nabla \pi.$$

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3. Non-linearity in NSE as right-hand side. Variation of constants formula:
$$u(t) = \mathrm{e}^{-tA} a - \int_0^t \mathrm{e}^{-(t-s)A}\, \mathbb{P}(u(s) \cdot \nabla) u(s) \mathrm{d}x.$$

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4. Kato iteration:
$$u_{j+1}(t) := u_0(t) - \int_0^t \mathrm{e}^{-(t-s)A}\, \mathbb{P}(u_j(s) \cdot \nabla) u_j(s) \mathrm{d}s.$$

# On Resolvent Estimates in $L^p$ for the Stokes Operator in Lipschitz Domains

Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$, a bounded Lipschitz domain.

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**Theorem (Fabes, Mendez, Mitrea, 1998)**

*There is $\varepsilon = \varepsilon(\Omega, d) > 0$, s.t. for all $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$ the Helmholtz projection exists on $L^p(\Omega; \mathbb{C}^d)$. Furthermore, the projection*

$$\mathbb{P} \colon L^p(\Omega; \mathbb{C}^d) \to L^p(\Omega; \mathbb{C}^d)$$

*is continuous.*

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## Theorem (Shen, 2012)

*Let $\theta \in [0, \pi)$. There exists $\varepsilon(\theta, d, \Omega) > 0$, s.t. for all*

$$\frac{2d}{d+1} - \varepsilon < p < \frac{2d}{d-1} + \varepsilon$$

*the Stokes operator $A_p$ on $\mathrm{L}^p_\sigma(\Omega)$ is sectorial with angle $\theta$. $A_p$ is closed, densely defined, $0 \in \rho(A_p)$ and $-A_p$ generates an analytic semigroup.*

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- Help of my supervisor: Dr. Patrick Tolksdorf

# At TUHH

Research

- Limit Operator Method
- Fibonacci Hamiltonian
- ...

Teaching

- Linear Algebra (Marko Lindner)
- Mathematical Image Processing (Marko Lindner)
- ZLL
- ...